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A STUDY OF CAPUTO-HADAMARD FRACTIONAL VOLTERRA-FREDHOLM INTEGRO-DIFFERENTIAL EQUATIONS WITH NONLOCAL BOUNDARY CONDITIONS

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Abstract. In this study, the Volterra-Fredholm equation which is a nonlinear integro-differential equation is discussed. In the first stage, the integro-differential equation was extended to the Volterra-Fredholm integro-differential equations involving the recently explored Caputo-Hadamard fractional derivatives. After, existence and uniqueness of positive solutions were obtained to such equations in Banach spaces via fixed point techniques and the method of upper and lower solutions. Finally, an illustrative example was considered for the extended problem by using the Caputo-Hadamard fractional derivative via fixed point technique.

1. Introduction

Many nonlinear differential equations are used to describe real world problems. To describe complex problems, the concept of a fractional-order derivative and a differential equation are used. Fractional Differential Equations (FDEs) with and without delay arise from a variety of applications including in various fields of science and engineering such as engineering technique fields, applied sciences, practical problems concerning mechanics, physics, dynamics, economy, control systems, chemistry, atomic energy, biology, medicine, information theory, harmonic oscillator, nonlinear oscillations, conservative systems, stability and instability of geodesic on Riemannian manifolds, dynamics in Hamiltonian systems, etc. In particular, problems concerning qualitative analysis of linear and nonlinear FDEs with and without delay have received the attention of many authors, see [11,21,24,25,27,29,32,33] and the references therein.

The fractional derivative of hadamard type introduced by Hadamard in 1892, differs from the Caputo and Riemann-Liouville derivatives in the sense that the kernel of the

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integral contains a logarithmic function of arbitrary exponent [1–3, 5]. Recently, the study of Hadamard FDEs is also of great importance. There has been a significant development in Hadamard derivative of differential equations in recent years for detail study on Hadamard fractional derivative, we refer to [6, 7, 9, 23, 30, 31].

Lately, there has been a developing interest for the fractional integro-differential equations (FIDEs). FIDEs have been recently used as effective tools in the modeling of many phenomena in various fields of applied sciences and engineering such as acoustic control, signal processing, porous media, electrochemistry, viscoelasticity, rheology, polymer physics, proteins, electromagnetics, optics, medicine, economics, astrophysics, chemical engineering, chaotic dynamics, statistical physics and so on [8, 10, 12, 13, 17, 19–22, 25, 29, 32, 33]. Many problems can be modeled by FIDE from various sciences and engineering applications.

Zhang in [36] investigated the existence and uniqueness of positive solutions for the nonlinear FDE

$$\begin{aligned} &D^{\nu}u(t) = f(t, u(t)),\ t \in (0, 1],\ 0 < \nu < 1, \\ &u(0) = 0, \end{aligned}$$

where D^{ν} is the standard Riemann-Liouville fractional derivative of order ν and $f : [0, 1] \times [0, \infty) \to [0, \infty)$ is a given continuous function. By using the method of the upper and lower solution and cone fixed point theorem, the author obtained the existence and uniqueness of a positive solution.

In [28], Matar discussed the existence and uniqueness of the positive solution of the following nonlinear FDE

$$\begin{aligned} &{}^{c}D^{\nu}u(t) = f(t, u(t)),\ t \in (0, 1],\ 0 < \nu \leq 2, \\ &u(0) = 0, u'(0) = \Phi > 0, \end{aligned}$$

where ${}^{c}D^{\nu}$ is the standard Caputo's fractional derivative of order ν and $f : [0, 1] \times [0, \infty) \to [0, \infty)$ is a given continuous function. By employing the method of the upper and lower solutions and Schauder and Banach fixed point theorems, the author obtained positivity results.

Ardjouni and Djoudi in [7] studied the positivity of the solutions for the nonlinear FDE with integral boundary conditions

$$\begin{aligned} &{}^{H}D^{\nu}u(t) = f(t, u(t)),\ t \in [1, T],\ 0 < \nu \leq 1, \\ &u(1) = u_0 + \lambda \int_1^T u(s)ds, \end{aligned}$$

where ${}^{H}D^{\nu}$ is the Caputo-Hadamard fractional derivative of order ν, $\lambda \geq 0, u_0 > 0$, $f : [1, T] \times [0, \infty) \to [0, \infty)$ is a given continuous function. By using the method of the upper and lower solution and Schauder and Banach fixed point theorems, the author obtained the existence and uniqueness of a positive solution.

In this paper, we extend the results in [6, 7] by proving the positivity of solutions for the following nonlinear Caputo-Hadamard fractional Volterra-Fredholm integro-differential equation

$$ {}^{H}D_{1}^{\nu}u(t) = f(t,u(t)) + \int_{1}^{t} k(t,s,u(s))ds + \int_{1}^{T} h(t,s,u(s))ds, \ t \in J := [1,T], \quad (1.1) $$

$$ u(1) = u_0 + \lambda \int_{1}^{T} u(s)ds, \quad (1.2) $$

where ${}^{H}D_{1}^{\nu}$ is the Caputo-Hadamard fractional derivative of order ν, $0 < \nu < 1$, $\lambda \geq 0, u_0 > 0$, $f : [1,T] \times [0,\infty) \to [0,\infty)$ and $k, h : [1,T] \times [1,T] \times [0,\infty) \to [0,\infty)$ are given continuous functions, k, h are non-decreasing on u. To prove the existence and uniqueness of positive solutions, we transform (1.1) into an equivalent integral equation and then by use the Krasnoselskii and Banach fixed point theorems.

This paper is organized as follows. In section 2, we introduce some notations and lemmas, and state some preliminaries results needed in later section. Also, we present the inversion of (1.1) and the Banach and Schauder fixed point theorems. In section 3, we give and prove our main results on positivity. In section 4, we provide an example to illustrate our results. In section 5, concluding remarks close the paper.

2. Preliminaries

Let $X = C(J)$ be the Banach space of all real-valued continuous functions defined on the compact interval J, endowed with the maximum norm. Define the the subspace $E = \{u \in X : u(t) \geq 0, \forall t \in J\}$.

Let us first recall some basic definitions, propositions and lemmas, which will be used throughout the work. For more details, see [1, 14–16, 18, 27, 28, 35].

Definition 2.1. [1, 27] The Hadamard derivative of fractional order $\nu > 0$ for a continuous function $h : [1,\infty) \longrightarrow \mathbb{R}$ is defined as

$$ D^{\nu}h(t) = \frac{1}{\Gamma(n-\nu)}\Big(t\frac{d}{dt}\Big)^{n} \int_{1}^{t} \Big(\log\frac{t}{s}\Big)^{n-\nu-1} h(s)\frac{ds}{s}, \ n-1 < \nu < n. \quad (2.1) $$

where $n = [\nu] + 1$, and $[\nu]$ denotes the integer part of real number ν and $\log(.) = \log_e(.)$.

Definition 2.2. [1] The Hadamard fractional integral of order ν for a continuous function h is defined as

$$ I^{\nu}h(t) = \frac{1}{\Gamma(\nu)} \int_{1}^{t} \Big(\log\frac{t}{s}\Big)^{\nu-1} h(s)\frac{ds}{s}, \ \nu > 0, $$

provided the integral exists.

Definition 2.3. [35] The Riemann-Liouville fractional integral of order $\nu > 0$ of a function f is defined as

$$ \begin{aligned} J^{\nu}h(t) &= \frac{1}{\Gamma(\nu)} \int_{0}^{t} (t-s)^{\nu-1}h(s)ds, \qquad t > 0, \quad \nu \in \mathbb{R}^{+}, \\ J^{0}h(t) &= h(t), \end{aligned} \quad (2.2) $$

where $\mathbb{R}^{+}$ is the set of positive real numbers.

Definition 2.4. [27] The Riemann-Liouville derivative of order ν with the lower limit zero for a function $h : [0, \infty) \longrightarrow \mathbb{R}$ can be written as

$$D^{\nu}h(t) = \frac{1}{\Gamma(1-\nu)}\frac{d}{dt}\int_0^t \frac{h(s)}{(t-s)^{\nu}}ds, \ t > 0, \ 0 < \nu < 1. \tag{2.3}$$

Definition 2.5. [34] Let $a, b \in \mathbb{R}^+$ and $b > a$. For any $u \in [a, b]$, we define the upper-control function $U(t, u) = \sup_{a \le \beta \le u} f(t, \beta)$ and lower-control function $L(t, u) = \inf_{u \le \beta \le b} f(t, \beta)$. Obviously, $U(t, u)$ and $L(t, u)$ are monotonous non-decreasing on u and

$$L(t, u) \le f(t, u) \le U(t, u).$$

Lemma 2.1. [4] *Let $n - 1 < \nu \le n, n \in \mathbb{N}$ and $u \in C^n([J])$. Then*

$$(I_1^{\nu} D_1^{\nu} u)(t) = u(t) - \sum_{k=0}^{n-1} \frac{u^(k)(1)}{\Gamma(k+1)}(\log t)^k.$$

Lemma 2.2. [4] *For all $\mu > 0$ and $\alpha > -1$*

$$\frac{1}{\Gamma(\mu)}\int_1^t \Big(\log\frac{t}{s}\Big)^{\mu-1}(\log s)^{\alpha}\frac{ds}{s} = \frac{\Gamma(\alpha+1)}{\Gamma(\mu+\alpha+1)}(\log t)^{\mu+\alpha}.$$

Theorem 2.1. [35] *(Banach's fixed point theorem) Let (X, d) be a nonempty complete metric space with $T : X \longrightarrow X$ is a contraction mapping. Then map T has a fixed point $x^* \in X$ such that $Tx^* = x^*$.*

Theorem 2.2. [35] *(Schauder's fixed point theorem) Let X be a Banach space and $B \subset X$ be a convex, closed and bounded set. If $\Omega : B \longrightarrow B$ is a continuous operator such that $\Omega B \subset X$, ΩB is relatively compact, then Ω has at least one fixed point in B.*

3. Existence and uniqueness results

In this section, we shall give the existence and uniqueness results of Eq.(1.1), with the conditions (1.2) and prove it. Before starting and proving the main results, we introduce the following lemma.

Lemma 3.1. [7,26] *Let $0 < \nu < 1$. Assume that $u \in C^1([1, T])$. Then u satisfies the problem (1.1)-(1.2) if and only if u satisfies the mixed type integral equation*

$$\begin{aligned} u(t) &= \frac{1}{\Gamma(\nu)}\int_1^t \Big(\log\frac{t}{\tau}\Big)^{\nu-1}\Big[f(\tau, u(\tau)) + \int_1^{\tau} k(\tau, s, u(s))ds + \int_1^T h(\tau, s, u(s))ds\Big]\frac{d\tau}{\tau} \\ &\quad + u_0 + \lambda\int_1^T u(\tau)d\tau, \ t \in J. \end{aligned} \tag{3.1}$$

Proof. Suppose u satisfies the problem (1.1)-(1.2), then applying I_1^{ν} to both sides of (1.1), we get

$$I_1^{\nu} D_1^{\nu} u(t) = I_1^{\nu}\Big(f(\tau, u(\tau)) + \int_1^{\tau} k(\tau, s, u(s))ds + \int_1^T h(\tau, s, u(s))ds\Big).$$

By using Lemma 2.1 and the integral boundary condition, we obtain

$$u(t) = \frac{1}{\Gamma(\nu)}\int_1^t \Big(\log\frac{t}{\tau}\Big)^{\nu-1}\Big[f(\tau,u(\tau)) + \int_1^\tau k(\tau,s,u(s))ds + \int_1^T h(\tau,s,u(s))ds\Big]\frac{d\tau}{\tau} + u_0 + \lambda\int_1^T u(\tau)d\tau,\ t\in J. \tag{3.2}$$

Conversely, suppose u satisfies (3.1), then applying D_1^ν to both sides of (3.1), we obtain

$$\begin{aligned} D_1^\nu u(t) &= D_1^\nu\Big(\frac{1}{\Gamma(\nu)}\int_1^t \Big(\log\frac{t}{\tau}\Big)^{\nu-1}\Big[f(\tau,u(\tau)) + \int_1^\tau k(\tau,s,u(s))ds + \int_1^T h(\tau,s,u(s))ds\Big]\frac{d\tau}{\tau} \\ &\quad + u_0 + \lambda\int_1^T u(\tau)d\tau\Big) \\ &= D_1^\nu I_1^\nu\Big(f(t,u(t)) + \int_1^t k(t,s,u(s))ds + \int_1^T h(t,s,u(s))ds\Big) + D_1^\nu\Big(u_0 + \lambda\int_1^T u(\tau)d\tau\Big) \\ &= f(t,u(t)) + \int_1^t k(t,s,u(s))ds + \int_1^T h(t,s,u(s))ds \end{aligned}$$

Moreover, the integral boundary condition $u(1) = u_0 + \lambda\int_1^T u(s)ds$, holds. □

To transform (3.2) to be applicable to Schauder's fixed point, we define the operator $\Omega : B \longrightarrow B$ by

$$(\Omega u)(t) = \frac{1}{\Gamma(\nu)}\int_1^t \Big(\log\frac{t}{\tau}\Big)^{\nu-1}\Big[f(\tau,u(\tau)) + \int_1^\tau k(\tau,s,u(s))ds + \int_1^T h(\tau,s,u(s))ds\Big]\frac{d\tau}{\tau} + u_0 + \lambda\int_1^T u(\tau)d\tau,\ t\in J, \tag{3.3}$$

where figured fixed point must satisfy the identity operator equation $\Omega u = u$. We introduce the following hypotheses:

(A1) Let $u^*, u_* \in B$ such that $a \le u_*(t) \le u^*(t) \le b$ and

$$D_1^\nu u^*(t) - \int_1^t k(t,s,u^*(s))ds - \int_1^T h(t,s,u^*(s))ds \ge U(t,u^*(t)),$$
$$D_1^\nu u_*(t) - \int_1^t k(t,s,u_*(s))ds - \int_1^T h(t,s,u_*(s))ds \le L(t,u_*(t)),\ \text{for any } t\in J.$$

(A2) There exist three positive constants L_f, L_k and L_h such that

$$\begin{aligned} &|f(t,u) - f(t,v)| \le L_f|u-v| \\ &|k(t,s,u) - k(t,s,v)| \le L_k|u-v|, \\ &|h(t,s,u) - h(t,s,v)| \le L_h|u-v|,\ \forall t,s\in J \text{ and } u,v\in\mathbb{R}. \end{aligned}$$

The functions u^* and u_* are respectively called the pair of upper and lower solutions for the problem (1.1)-(1.2).

The first result is based on the Schauder fixed point theorem.

Theorem 3.1. *Assume that the hypothesis (A1)-(A2) are fulfilled, then there exists at least one positive solution for the problem (1.1)-(1.2).*

Proof. Let $\Phi = \{u \in B : u_*(t) \leq u(t) \leq u^*(t), t \in J\}$ endowed with the norm $\|u\| = \max_{t\in J}|u(t)|$, then we have $\|u\| \leq b$. Hence, Φ is convex bounded and closed subset of the Banach space $C([1,T])$. Moreover, the continuity of f, k and h imply the continuity of the operator Ω on Φ defined by (3.3). Now, if $u \in \Phi$, there exist three positive constants M_f, M_k and M_h such that

$$\max\{f(t,u(t)) : t \in J, u(t) \leq b\} \leq M_f,$$
$$\max\{k(t,s,u(s)) : t, s \in J, u(s) \leq b\} \leq M_k,$$

and

$$\max\{h(t,s,u(s)) : t, s \in J, u(s) \leq b\} \leq M_h.$$

Then

$$\begin{aligned}
|(\Omega u)(t)| &\leq \frac{1}{\Gamma(\nu)}\int_1^t \Big(\log\frac{t}{\tau}\Big)^{\nu-1}\Big[|f(\tau,u(\tau))| + \int_1^\tau |k(\tau,s,u(s))|ds + \int_1^T |h(\tau,s,u(s))|ds\Big]\frac{d\tau}{\tau} \\
&\quad + u_0 + \lambda\int_1^T |u(\tau)|d\tau \\
&\leq \frac{M_f(\log T)^\nu}{\Gamma(\nu+1)} + \frac{M_k(\log T)^{\nu+1}}{\Gamma(\nu+2)} + \frac{M_h(\log T)^{\nu+1}}{\Gamma(\nu+2)} + u_0 + \lambda b(T-1).
\end{aligned}$$

Thus

$$\|\Omega u\| \leq \frac{M_f(\log T)^\nu}{\Gamma(\nu+1)} + \frac{(M_k+M_h)(\log T)^{\nu+1}}{\Gamma(\nu+2)} + u_0 + \lambda b(T-1).$$

Hence, $\Omega(\Phi)$ is uniformly bounded. Next, we prove the equicontinuity of $\Omega(\Phi)$. For each $u \in \Phi$. Then for $t_1, t_2 \in J$ with $t_1 < t_2$, we have

$$\begin{aligned}
&|(\Omega u)(t_2) - (\Omega u)(t_1)| \\
\leq\ & \frac{1}{\Gamma(\nu)}\int_1^{t_1}\Big[\Big(\log\frac{t_1}{\tau}\Big)^{\nu-1} - \Big(\log\frac{t_2}{\tau}\Big)^{\nu-1}\Big]|f(\tau,u(\tau))|\frac{d\tau}{\tau} \\
&+ \frac{1}{\Gamma(\nu)}\int_{t_1}^{t_2}\Big(\log\frac{t_2}{\tau}\Big)^{\nu-1}|f(\tau,u(\tau))|\frac{d\tau}{\tau} \\
&+ \frac{1}{\Gamma(\nu)}\int_1^{t_1}\Big[\Big(\log\frac{t_1}{\tau}\Big)^{\nu-1} - \Big(\log\frac{t_2}{\tau}\Big)^{\nu-1}\Big]\Big(\int_1^s |k(\tau,s,u(s))|ds + \int_1^T |h(\tau,s,u(s))|ds\Big)\frac{d\tau}{\tau} \\
&+ \frac{1}{\Gamma(\nu)}\int_{t_1}^{t_2}\Big(\log\frac{t_2}{\tau}\Big)^{\nu-1}\Big(\int_1^s |k(\tau,s,u(s))|ds + \int_1^T |h(\tau,s,u(s))|ds\Big)\frac{d\tau}{\tau} \\
\leq\ & \frac{M_f}{\Gamma(\nu+1)}\Big[2(\log\frac{t_2}{t_1})^\nu + (\log t_1)^\nu - (\log t_2)^\nu\Big] \\
&+ \frac{(M_k+M_h)}{\Gamma(\nu+2)}\Big[2(\log\frac{t_2}{t_1})^{\nu+1} + (\log t_1)^{\nu+1} - (\log t_2)^{\nu+1}\Big] \\
\leq\ & \frac{2M_f}{\Gamma(\nu+1)}\Big(\log\frac{t_2}{t_1}\Big)^\nu + \frac{2(M_k+M_h)}{\Gamma(\nu+2)}\Big(\log\frac{t_2}{t_1}\Big)^{\nu+1} \\
&\longrightarrow 0 \text{ as } t_1 \longrightarrow t_2.
\end{aligned} \tag{3.4}$$

The convergence is independent of u in Φ, which means that $\Omega(\Phi)$ is equicontinuous. The Arzela-Ascoli theorem implies that $\Omega : \Phi \longrightarrow B$ is compact. The only thing to apply the Schauder fixed point is to prove that $\Omega(\Phi) \subset \Phi$. For any $u \in \Phi$, then $u_*(t) \leq u(t) \leq u^*(t)$ and by (A1), we have

$$\begin{aligned}
(\Omega u)(t) &= \frac{1}{\Gamma(\nu)} \int_1^t \Big(\log \frac{t}{\tau}\Big)^{\nu-1} \Big[f(\tau, u(\tau)) + \int_1^\tau k(\tau, s, u(s))ds + \int_1^T h(\tau, s, u(s))ds\Big] \frac{d\tau}{\tau} \\
&\quad + u_0 + \lambda \int_1^T u(\tau)d\tau \\
&\leq \frac{1}{\Gamma(\nu)} \int_1^t \Big(\log \frac{t}{\tau}\Big)^{\nu-1} \Big[U(\tau, u(\tau)) + \int_1^\tau k(\tau, s, u(s))ds + \int_1^T h(\tau, s, u(s))ds\Big] \frac{d\tau}{\tau} \\
&\quad + u_0 + \lambda \int_1^T u(\tau)d\tau \\
&\leq \frac{1}{\Gamma(\nu)} \int_1^t \Big(\log \frac{t}{\tau}\Big)^{\nu-1} \Big[U(\tau, u^*(\tau)) + \int_1^\tau k(\tau, s, u^*(s))ds + \int_1^T h(\tau, s, u^*(s))ds\Big] \frac{d\tau}{\tau} \\
&\quad + u_0 + \lambda \int_1^T u^*(\tau)d\tau \\
&\leq u^*(t),
\end{aligned}$$

and

$$\begin{aligned}
(\Omega u)(t) &= \frac{1}{\Gamma(\nu)} \int_1^t \Big(\log \frac{t}{\tau}\Big)^{\nu-1} \Big[f(\tau, u(\tau)) + \int_1^\tau k(\tau, s, u(s))ds + \int_1^T h(\tau, s, u(s))ds\Big] \frac{d\tau}{\tau} \\
&\quad + u_0 + \lambda \int_1^T u(\tau)d\tau \\
&\geq \frac{1}{\Gamma(\nu)} \int_1^t \Big(\log \frac{t}{\tau}\Big)^{\nu-1} \Big[L(\tau, u(\tau)) + \int_1^\tau k(\tau, s, u(s))ds + \int_1^T h(\tau, s, u(s))ds\Big] \frac{d\tau}{\tau} \\
&\quad + u_0 + \lambda \int_1^T u(\tau)d\tau \\
&\geq \frac{1}{\Gamma(\nu)} \int_1^t \Big(\log \frac{t}{\tau}\Big)^{\nu-1} \Big[L(\tau, u_*(\tau)) + \int_1^\tau k(\tau, s, u_*(s))ds + \int_1^T h(\tau, s, u_*(s))ds\Big] \frac{d\tau}{\tau} \\
&\quad + u_0 + \lambda \int_1^T u_*(\tau)d\tau \\
&\geq u_*(t).
\end{aligned}$$

Hence, $u_*(t) \leq (\Omega u)(t) \leq u^*(t),\ t \in J$, that is, $\Omega(\Phi) \subset \Phi$. According to the Schauder fixed point theorem, the operator Ω has at least one fixed point $u \in \Phi$. Therefore, the problem (1.1)-(1.2) has at least one positive solution, and the proof is completed. □

The second result is based on the Banach fixed point theorem.

Theorem 3.2. *Assumes that (A1) and (A2) hold, and if*

$$\Delta := \frac{L_f\Big(\log T\Big)^\nu}{\Gamma(\nu+1)} + \frac{(L_k + L_h)\Big(\log T\Big)^{\nu+1}}{\Gamma(\nu+2)} + \lambda(T-1) < 1. \tag{3.5}$$

Then the problem (1.1)-(1.2) *has a unique positive solution.*

Proof. From Theorem 3.1, it follows that the problem (1.1)-(1.2) has at least one positive solution. Hence, we need only to prove that the operator defined in (3.3) is a contraction in Φ. In fact, for any $u, v \in \Phi$, we have

$$\begin{aligned}
&|(\Omega u)(t) - (\Omega v)(t)| \\
\leq\ & \frac{1}{\Gamma(\nu)} \int_1^t \Big(\log\frac{t}{\tau}\Big)^{\nu-1} \Big|f(\tau, u(\tau)) - f(\tau, v(\tau))\Big| \frac{d\tau}{\tau} \\
&+ \frac{1}{\Gamma(\nu)} \int_1^t \Big(\log\frac{t}{\tau}\Big)^{\nu-1} \Big[\int_1^\tau \Big|k(\tau, s, u(s)) - k(\tau, s, v(s))\Big| ds \\
&+ \int_1^T \Big|h(\tau, s, u(s)) - h(\tau, s, v(s))\Big| ds\Big] \frac{d\tau}{\tau} + \lambda \int_1^T |u(\tau) - v(\tau)| d\tau \\
\leq\ & \Big(\frac{L_f\Big(\log T\Big)^{\nu}}{\Gamma(\nu+1)} + \frac{(L_k + L_h)\Big(\log T\Big)^{\nu+1}}{\Gamma(\nu+2)} + \lambda(T-1)\Big)\|u - v\|.
\end{aligned}$$

Thus

$$\|\Omega u - \Omega v\| \leq \Delta \|u - v\|.$$

Hence, the operator Ω is a contraction mapping by inequality (3.5). Therefore, by the Banach fixed point theorem, we conclude that the problem (1.1)-(1.2) has a unique positive solution. □

4. An example

Example 1. Consider the fractional Volterra-Fredholm integro-differential equation with integral boundary conditions

$$^{H}D_1^{\frac{2}{3}} u(t) = \frac{1}{2}(3 + \cos(u(t))) + \frac{1}{6}\int_1^t u(s) e^{-(t^2+s^2)} ds + \frac{1}{6}\int_1^e u(s) e^{-s^2} ds \tag{4.1}$$

$$u(1) = \frac{3}{2} + \frac{1}{6}\int_1^e u(s) ds \tag{4.2}$$

where $T = e, \nu = \frac{2}{3}, \lambda = \frac{1}{6}, u_0 = \frac{3}{2}, f(t, u(t)) = \frac{1}{2}(3 + \cos(u(t))), k(t, s, u(s)) = u(s)e^{-(t^2+s^2)}$ and $h(t, s, u(s)) = u(s)e^{-s^2}$. Since f is continuous positive functions, k and h are nondecreasing on u and

$$\frac{L_f\Big(\log T\Big)^{\nu}}{\Gamma(\nu+1)} + \frac{(L_k + L_h)\Big(\log T\Big)^{\nu+1}}{\Gamma(\nu+2)} + \lambda(T-1) \simeq 0.65 < 1,$$

then, by Theorem 3.2, the problem (4.1)-(4.2) has a unique positive solution.

5. Conclusions

We can conclude that the main results of this article have been successfully achieved, that is, through of Banach and Schauder's fixed point theorems, we have investigated the existence and uniqueness of positive solutions of a nonlinear Caputo-Hadamard fractional Volterra-Fredholm integro-differential equation.

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