Turkish J. Ineq., 5 (1) (2021), Pages 16 –39 .

Turkish Journal of
INEQUALITIES

Available online at www.tjinequality.com

A TWO STEP EXTRAGRADIENT-VISCOSITY ALGORITHM FOR COMMON FIXED POINT PROBLEM OF TWO ASYMPTOTICALLY NONEXPANSIVE MAPPINGS AND VARIATIONAL INEQUALITY PROBLEMS IN BANACH SPACES

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Abstract. ln this paper, we propose a two-step extragradient-viscosity iteration algorithm for finding common element of the set of solutions of the variational inequality problem for accretive mappings and the set of fixed points of two asymptotically nonexpansive mappings in the framework of uniformly convex Banach space and 2-uniformly smooth Banach space. In addition, we prove strong convergence theorems of the proposed iterative algorithm. Finally, we prove that a slight modification of our new scheme could be employed in solving variational inequality problems in Hilbert space. Our results improve, extend and generalize several known results in literature.

1. Introduction

Throughout this paper, we assume, unless otherwise specified, that C is a nonempty, closed and convex subset of a Banach space E whose dual space is $E^\star$, $D(T)$ and $R(T)$ are the domain and range of the mapping T, $N, R, R^+, \rightarrow$ and $\rightharpoonup$ will denote the set of natural numbers, the set of real numbers, the set of nonnegative real numbers, strong convergence and weak convergence respectively. In what follows, the mapping $J : E \longrightarrow 2^{E^\star}$ defined by

$$J(x) = \{x^\star \in E : \langle x, x^\star\rangle = \|x\|\|x^\star\|, \|x\| = \|x^\star\|\}, \tag{1.1}$$

is called normalised duality mapping, where $\langle ., .\rangle$ denotes the generalized duality pairing of elements between E and $E^\star$. It is well known that E is smooth if and only if J is single-valued, uniformly smooth if and only if each duality map J is norm-to-norm uniformly continuous on bounded subset of E. (see [10], [27] for more details on the duality mapping and its properties).

Key words and phrases. Strong convergence, Variational inequality, Extragradient-Viscosity algorithm, Asymptotically nonexpansive mapping, Common fixed point, Banach space.

2010 *Mathematics Subject Classification.* Primary: 39B82. Secondary: 44B20, 46C05.

Received: 06/12/2020 *Accepted*: 15/04/2021.

Cited this article as: I.K. Agwu, D.I. Igbokwe, A Two Step Extragradient-Viscosity Algorithm for Common Fixed Point Problem of Two Asymptotically Nonexpansive Mappings and Variational Inequality Problems in Banach Spaces, Turkish Journal of Inequalities, 5(1) (2021), 16-39.

A Banach space is said to have a weakly continuous duality map, in the sense of Browder [9], if there exists a gauge function $\phi : [0, \infty) \longrightarrow [0, \infty)$ with $\phi(0) = 0$ such that the duality map J with the guage function ϕ is single-valued and is weak-to-weak* sequentially continuous; that is, if $\{x_n\} \subset E, x_n \overset{w}{\to} x$, then $J_\phi(x_n) \overset{w^\star}{\to} J_\phi(x)$. It is known that $\ell^p(1 < p < \infty)$ has a weakly continuous duality map with guage function $\phi(t) = t^{p-1}$, see for example [10] for more details.

Let C nonempty subset of a real Banch spce E. Let $S : C \longrightarrow C$ be a mapping, then we denote the set of fixed point of S by $F(S)$. Let $S, T : C \longrightarrow C$ be two given nonlinear mappings. The set of common fixed point of the two mappings S and T will be denoted by $\mathcal{F} = F(S) \cap F(T)$.

Definition 1.1. Recall that a nonlinear mapping T is said to be:

(a) Lipschitizian if there exists a constant L such that

$$\|Tx - Ty\| \leq L\|x - y\|, \forall x, y \in D(T), \tag{1.2}$$

where L is the Lipschitizian constant of T. Note that T is contraction if $L \in (0, 1)$ in (1.2) and nonexpansive if $L = 1$ in (1.2).

(b) uniformly Lipschitizian if for all $n \in N$, there exists a constant L such that

$$\|T^n x - T^n y\| \leq L\|x - y\|, \forall x, y \in D(T). \tag{1.3}$$

(c) asymptotically nonexpansive [1] if for all $x, y \in D(T)$ and $n \in \mathbb{N}$, there exists a sequence $k_n \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that the following inequality:

$$\|T^n x - T^n y\| \leq k_n\|x - y\| \tag{1.4}$$

holds.

Remark 1.1. If the classes of mappings such as nonexpansive, asymptotically nonexpansive, uniformly Lipschitizian and Lipschitizian are represented by $(N), (AN), (UL)$ and (L), respectively, then the following relationship

$$(N) \subset (AN) \subset (UL) \subset (L) \tag{1.5}$$

is evident (see [49] for details).

Since the emergence of the equation of the type (1.6), researchers in mathematics and mathematical sciences have discovered that many practical problems arising from different areas of optimisation, engineering, variational inequalities, differential equations, mathematical sciences can be modeled by the equation of the form:

$$x = Tx, \tag{1.6}$$

where T is a nonexpansive mapping. The solution set of the problem defined by (1.6) coincides with the fixed point set of T. In about forty (40) years or so, some researchers have studied the type of operator defined in (1.6) in the context of different mappings and different underlying spaces, and many more are still deeply involved in some investigations to learn more about some generations and practical implications of its inherent properties (see, for example, [3–6, 12, 13, 17, 19, 23, 28, 31, 32, 35, 38, 42, 52–54] for more details).

Definition 1.2. Let C be a nonempty subset of a real Banach space E. Recall that an operator $A : C \longrightarrow E$ is called :

(a) accretive if there exists $j(x-y) \in J(X-Y)$ such that

$$\langle Ax - Ay, j(x-y)\rangle \geq 0, \forall x, y \in C. \tag{1.7}$$

(b) α-inverse strongly accretive if for some $\alpha > 0$

$$\langle Ax - Ay, j(x-y)\rangle \geq \alpha\|x-y\|, \forall x, y \in C. \tag{1.8}$$

It is worthy to note that in Hilbert spaces, the normalized duality map is the identity map. Hence, in Hilbert space, accretivity and monotonicity coincide.

In solving variational inequality problems, Aoyama et al [5], Ceng et al [53], Yao et al [45] and Cai, Shehu and Iyiola [6] considered the following problems for accretive and α-strongly operators:

(a) find a point $x^\star \in C$ such that, for some $j(x-x^\star) \in J(x-x^\star)$ such that

$$\langle Ax^\star, j(x-x^\star)\rangle \geq 0, \forall x \in C. \tag{1.9}$$

The solution set of (1.9) is denoted by $VI(C, A)$; that is,

$$VI(C, A) = \{x^\star \in C : \langle Ax^\star, j(x-x^\star)\rangle \geq 0, \forall x \in C\}.$$

(b) find $(x^\star, y^\star) \in C \times C$ such that

$$\begin{cases} \langle \lambda Ay^\star + x^\star - y^\star, x - x^\star\rangle \geq 0, \forall x \in C, \\ \langle \mu Bx^\star + y^\star - x^\star, x - y^\star\rangle \geq 0, \forall x \in C, \end{cases} \tag{1.10}$$

which is called general system of variational inequality, where $A, B : C \longrightarrow H$ are nonlinear mappings and $\lambda, \mu > 0$ are two constants.

(c) find $(x^\star, y^\star) \in C \times C$ such that

$$\begin{cases} \langle Ay^\star + x^\star - y^\star, j(x - x^\star)\rangle \geq 0, \forall x \in C, \\ \langle Bx^\star + y^\star - x^\star, j(x - y^\star)\rangle \geq 0, \forall x \in C, \end{cases} \tag{1.11}$$

where $A, B : C \longrightarrow H$ are nonlinear mappings.

(d) find $(x^\star, y^\star) \in C \times C$ such that

$$\begin{cases} \langle \lambda Ay^\star + x^\star - y^\star, j(x - x^\star)\rangle \geq 0, \forall x \in C, \\ \langle \mu Bx^\star + y^\star - x^\star, j(x - y^\star)\rangle \geq 0, \forall x \in C, \end{cases} \tag{1.12}$$

where $A, B : C \longrightarrow H$ are nonlinear mappings and $\lambda, \mu > 0$ are two constants,

respectively. Observe that if $\lambda = 1 = \mu$, then (1.12) reduces to (1.11).

Current literature shows that some constraints of a good number of practical problems arising in image recovery, resource allocation, signal processing, etc can be expressed as the variational inequality problem. In line with this, a good number of established mathematicians are currently working on different ways of finding solutions of variational inequality problems (see, for example, [5–8, 21–26, 44–49, 51, 55, 56]).

In an attempt to solve the variational inequality problem of (1.9), Aoyama et al [5] studied the following algorithm:

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)Q_C(I-\lambda_n A)x_n, \tag{1.13}$$

where Q_C is sunny nonexpansive retraction from E onto C and $\alpha_n\} \subset (0,1)$ and $\{\lambda_n\} \subset (0,\infty)$ are real number sequences. They proved the following weak convergent result:

Theorem 1.1. (Aoyama et al. [5]) *Let C be a nonempty closed convex subset of 2-uniformly smooth and uniformly convex Banach space E. Let Q_C be the sunny nonexpansive retraction from E onto C. Let $A : C \longrightarrow E$ be an α-inverse-strongly accretive operator with $VI(C,A) \neq \emptyset$. If $\{\lambda_n\}$ and $\{\alpha_n\}$ are chosen so that $\lambda_n \in \left[a, \frac{\alpha}{K^2}\right]$ for some $a > 0$ and $\alpha_n \in [b,c]$ for some b, c with $0 < b < c < 1$, then the sequence $\{x_n\}$ defined by (1.13) converges weakly to z, a solution of the variational inequality (1.9), where the real number K is the 2-uniformly smoothness constant of the Banach space E.*

One of the most studied methods of approximating fixed points of nonexpansive mapping is the viscosity approximation method which was introduced by Moudafi [17]. Let C be a nonempty, closed and convex subset of a real Banach space E. Let $T : C \longrightarrow C$ be a nonexpansive mapping such that $F(T) \neq \emptyset$ and $f : C \longrightarrow C$ be a contraction mapping. The viscosity iteration method is defined as follows:
For $x_0 \in C$, let $\{x_n\}_{n\geq 1}$ be a sequence generated by

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)Tx_n, \tag{1.14}$$

where $\{\alpha_n\}_{n\geq 1}$ is a sequence of real numbers in $(0,1)$. Under appropriate conditions, the sequence defined by (1.14) converges to a fixed point of T.

In [6], Cai, Shehu and Iyiola introduced the following iterative scheme: Let C be a nonempty, closed subset of a real uniformly and $2-$uniformly smooth Banach space E. Let $T : C \longrightarrow C$ be an asymptotically nonexpansive mapping such that $F(T) \neq \emptyset$ and $f : C \longrightarrow C$ be a contraction mapping. Then extragradient-viscosity iteration method for the above mapping and problem (1.12) is defined as follows:

$$\begin{cases} x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n T^n z_n; \\ z_n = Q_C(I-\lambda A)u_n; \\ u_n = Q_C(I-\mu B)x_n, \end{cases} \tag{1.15}$$

Under suitable conditions on the iteration parameters, they proved strong convergence theorem of the sequence defined by (1.15) to common element of solution of the variational inequality problem (1.12) and fixed point problem of asymptotically nonexpansive mapping. To be precise, following theorem was proved by them:

Theorem 1.2. (Cai, Shehu, Iyiola [6]) *Let C be a nonempty closed convex subset of 2-uniformly smooth and uniformly convex Banach space X, which admits weakly sequentially continuous duality mapping. Assume that C is a sunny nonexpansive retract of X and let Q_C be the sunny nonexpansive retraction of X onto C. Let $A, B : C \longrightarrow X$ be α-inverse-strongly accretive and β-inverse-strongly accretive mappings, respectively. Let $f : C \longrightarrow C$*

be a δ-strict contraction of C into itself with coefficient $\rho \in (0,1)$. Let $T : C \longrightarrow C$ be asymptotically nonexpansive self mapping on C such that $\mathfrak{F} = F(T) \cap F(G) \neq \emptyset$, where G is as defined by Lemma 2.9. For arbitrarily chosen $x_1 \in C$, let the sequence $\{x_n\}_{n\geq 1}$ be defined iteratively as follows:

$$\begin{cases} x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n T^n z_n; \\ z_n = Q_C(I - \lambda A)u_n; \\ u_n = Q_C(I - \mu B)x_n, \end{cases}$$

where $0 < \lambda < \dfrac{\alpha}{K^2}, 0 < \mu < \dfrac{\beta}{K^2}$. Suppose $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are sequences in $[0,1]$ satisfying suitable conditions. Then, the sequence $\{x_n\}$ converges strongly to $q^\star = Q_F f(q)$ and $(q, q^\star)$ is a solution of problem (1.12), where $q^\star = Q_C(q - \mu Sq)$ Q_F is the sunny nonexpansive retraction of C onto F.

For work on extragradient algorithm for finding a common solution to split generalised mixed equilibrium problem, we refer the reader to [57].

Motivated by the above works, in this paper, we introduce and study the following algorithm for the class of two asymptotically quasi-nonexpansive mappings and the variational inequality problem (1.12) as follows: Let C be a nonempty, closed subset of a real uniformly convex and $2-$uniformly smooth Banach space E. Let $S, T : C \longrightarrow C$ be two asymptotically quasi-nonexpansive mappings such that $F(T) \neq \emptyset$ and $f : C \longrightarrow C$ be a contraction mapping. Then, the extragradient-viscosity iteration method for the above mappings and problem (1.12) is defined as follows:

$$\begin{cases} x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n - \gamma_n)x_n + \gamma_n T^n y_n; \\ y_n = \alpha'_n f(x_n) + (1 - \alpha'_n - \gamma'_n)x_n + \gamma'_n S^n z_n; \\ z_n = Q_C(I - \lambda A)u_n; \\ u_n = Q_C(I - \mu B)x_n, \end{cases} \tag{1.16}$$

where $0 < \lambda < \dfrac{\xi}{K^2}, 0 < \mu < \dfrac{\eta}{K^2}$, K is uniformly smoothness constant, $\xi, \eta > 0$, and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\alpha'_n\}, \{\beta'_n\}, \{\gamma'_n\}$ are sequences in $[0,1]$. Under suitable conditions on the control sequences, we prove strong convergence of the scheme defined by (1.16) to common element of solution of two asymptotically quasi-nonexpansive mapping and the variational inequality problem of (1.12).

Remark 1.2. The following remarks are evident from (1.16):

(1) If $\alpha' = 0, \gamma' = 1$, (1.16) reduces to:

$$\begin{cases} x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n - \gamma_n)x_n + \gamma_n S^n z_n; \\ z_n = Q_C(I - \lambda A)u_n; \\ u_n = Q_C(I - \mu B)x_n, \end{cases} \tag{1.17}$$

(2) If $B = I, \mu = 0, \alpha_n = 0 = \alpha'_n, \gamma'_n = 1$ and $S = I = T$, where I is the identity mapping, (1.16) reduces to:

$$x_{n+1} = (1-\gamma_n)x_n + \gamma_n Q_C(I - \lambda A)x_n \tag{1.18}$$

Note that (1.15) is not the same as (1.17) since the class of mapping T in (1.15) is a subclass of the class of mapping S in (1.17). Also, (1.18) is the same as (1.13). Consequently, the results presented in this paper extend, improve and generalize some recently announced results in the existing literature (see, for example, [4, 6–8, 13, 19, 30, 32, 33, 41–43, 46–48, 51] and the reference therein).

2. Preliminaries

For the sake of convenience, we restate the following concepts and results:
Let E be a Banach space with its dimension greater than or equal to 2. The modulus of convexity of E is a function $\delta_E(\varepsilon) : (0, 2] \longrightarrow (0, 2]$ defined by

$$\delta_E(\varepsilon) = \inf\{1 - \|\frac{1}{2}(x+y)\| : \|x\| = 1, \|y\| = 1, \varepsilon = \|x-y\|\}.$$

A Banach space E is uniformly convex if and if $\delta_E(\varepsilon) > 0$, for all $\varepsilon \in (0, 2]$.

Let E be a normed linear space and let $S = \{x \in E : \|x\| = 1\}$. E is called smooth if

$$\lim_{t\to 0^+} \frac{\|x+ty\| - \|x\|}{t}$$

exists for each $x, y \in S$. E is called uniformly smooth if it is smooth and the limit above is attained uniformly for each $x, y \in S$.

Let E be a normed space with dimension greater than or equal to 2. The modulus of smoothness of E is the function $\rho_E : [0, \infty) \longrightarrow [0, \infty)$ such that

$$\rho_E(\tau) = \sup\left\{\frac{\|x+y\| + \|x-y\|}{2} - 1 : \|x\| = 1, \|y\| = \tau\right\}.$$

It is known that a normed linear space E is uniformly smooth if

$$\lim_{\tau\to 0} \frac{\rho_E(\tau)}{\tau} = 0.$$

Note that if there exists a constant $c > 0$ and a real number $q > 1$ such that $\rho_E(\tau) \leq c\tau$, then E is called q-uniformly smooth. Typical examples of smooth spaces are L_p, ℓ_p and W_p^m for $1 < p < \infty$, where L_p, ℓ_p or W_p^m is 2-uniformly smooth and p-uniformly convex if $2 \leq p < \infty$; 2-uniformly convex and p-uniformly smooth if $1 < p < 2$.

Let D be a subset of C and let Q be a mapping of C into D. The Q is said to sunny if

$$Q(Qx + t(x - Qx)) = Qx, \tag{2.1}$$

whenever $Qx + t(x - Qx \in C$ and $t \geq 0$. A mapping Q of C into itself is called a retraction if $Q^2 = Q$. If mapping Q into itself is a retraction, then $Qz = z$ for every $z \in R(Q)$, where $R(Q)$ is the range of Q. A subset D of C is called a sunny nonexpansive retract of C if there exists a sunny nonexpansive retraction from C onto D. The following three lemmas 2.1, 2.2, 2.3 are known for sunny nonexpansive retraction:

Lemma 2.1. (see [24]) *Let C be a nonempty, closed and convex subset of a uniformly convex and uniformly smooth Banach space E and let T be a nonexpansive mapping of C into itself with $F(T) \neq \emptyset$. Then the set $F(T)$ is a sunny nonexpansive retraction on C.*

Lemma 2.2. (see [16]) *Let C be a nonempty closed convex subset of a smooth Banach space E and let Q_C be a retraction from E onto C. Let D be a nonempty subset of C. Let $Q : C \longrightarrow D$ be a retraction and J be a normalised duality map on E. Then, the following are equivalent:*

(i) *Q_C is both sunny and nonexpansive;*
(ii) $\|Qx - Qy\|^2 \leq \langle x - y.J(Ox - Qy)\rangle, \forall x, y \in C$;
(iii) $\langle x - Q_C x, J(y - Q_C x\rangle \leq 0, \forall x \in E$ and $y \in CD$.

It is well known that if E is a Hilbert space, then a sunny nonexpansive retraction Q_C coincides with the metric projection P_C from E onto C. Let C be a nonempty closed and convex subset of a smooth Banach space E, $x \in E$ and $x_0 \in C$. Then, we have from Lemma 2.2 that $x_0 \in Q_C x$ if and only if $\rangle x - x_0, J(y - x_0)\langle \leq 0, \forall y \in C$, where Q_C is a sunny nonexpansive retraction from E onto C.

Lemma 2.3. (see [5]) *Let C be a nonempty closed convex subset of a smooth Banach space E, Q_C be a sunny nonexpansive retraction from E onto C and A be an accretive operator of C into E. Then, for all $\lambda > 0$,*

$$VI(C, A) = Fix(Q_C(I - \lambda A)).$$

Proposition 2.1. (see [6]; see also [$Theorem \quad 4, 23$]) *Let D be a closed and convex subset of a refexive Banach space E with a uniformly Gateaux differentiable norm. If C is nonexpansive retract of D, then it in fact a sunny nonexpansive retract of D.*

Lemma 2.4. (see [43]) *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space E and let β_n be a sequence in $[0, 1]$, which satisfies the following condition: $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Suppose $x_{n+1} = \beta_n x_n + (1 - \beta_n)y_n, n \geq 0$ and $limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\| \leq 0$. Then, $\lim_{n\to\infty} \|y_n - x_n\| = 0$.*

Lemma 2.5. (see [19]) *Let $\{a_n\}$ be a sequence of nonnegative real numbers with $a_{n+1} = (1 - \alpha_n)a_n + b_n, n \geq 0$, where α_n is a sequence in $(0, 1)$ and b_n is a sequence in R such that $\sum_{n=0}^{\infty} = \infty$ and $\limsup_{n\to\infty} \dfrac{b_n}{\alpha_n} \leq 0$. Then, $\lim_{n\to\infty} a_n = 0$.*

Lemma 2.6. (see [25], [39]) *Let E be a real smooth and uniformly convex Banach space and $r > 0$. Them, there exists a strictly increasing, continuous and convex function $g : [0, 2r] \longrightarrow R$ with $g(0) = 0$ such that $g(\|x - y\| \leq \|x\|^2 - 2\langle x, j(y)\rangle + \|y\|^2$, for all $x, y \in B_r$.*

Lemma 2.7. (see [7]) *Let C be a nonempty closed convex subset of a real 2-uniformly smooth Banach space E. Let the mapping $A : C \longrightarrow E$ be an α-inverse-strongly accretive. Then, we have the following:*

$$\|(I - \lambda A)x - (I - \lambda A)y\|^2 \leq \|x - y\|^2 + 2\lambda(\lambda K^2 - \alpha)\|Ax - Ay\|^2,$$

where $\lambda > 0$. In particular, if $0 < \lambda \leq \dfrac{\alpha}{K^2}$, then $I - \lambda A$ is nonexpansive.

Lemma 2.8. (see [7]) *Let C be a nonempty closed convex subset of a real 2-uniformly smooth Banach space E. Assume that C is sunny nonexpansive retract of E and let Q_C be a sunny nonexpansive retraction of E onto C. Let the mappings $A, B : C \longrightarrow E$ be α-inverse-strongly accretive and β-inverse-strongly accretive respectively. Let $G : C \longrightarrow C$ be a mapping defined by*

$$G(x) = Q_C[Q_C(x - \mu Bx) - \lambda AQ_{C(x-\mu Bx)],\forall x\in C.}$$

If $0 < \lambda < \dfrac{\alpha}{K^2}$ and $0 < \mu < \dfrac{\beta}{K^2}$, then $G : C \longrightarrow C$ is nonexpansive.

Lemma 2.9. (see [34]) *Let C be a nonempty closed convex subset of a real 2-uniformly smooth Banach space E and let Q_C be a sunny nonexpansive retraction of E onto C. $A, B : C \longrightarrow E$ be two nonlinear mappings. For a given $x^\star, y^\star \in C, (x^\star, y^\star)$ is a solution to problem (1.18) if and only if $x^\star = Q_C(y^\star - \lambda Ay^\star)$, where $y^\star = Q_C(x^\star - \mu Bx^\star)$, that is $x^\star = Gx^\star$, where G is as defined by Lemma 2.9.*

Lemma 2.10. (see [34]) *Let E be a real Banach space and $J : E \longrightarrow 2^{E^\star}$ be a normalised duality mapping, then for any $x, y \in E$, the following inequalities hold:*

$$\begin{aligned} \|x+y\|^2 &\leq \|x\|^2 + 2\langle y, j(x+y)\rangle, \forall j(x+y) \in J(x+y); \\ \|x+y\|^2 &\geq \|x\|^2 + 2\langle y, j(x)\rangle, \forall j(x) \in J(x); \end{aligned}$$

Lemma 2.11. (see [10]) *Let C be a nonempty closed convex subset of a real uniformly convex Banach space E and let T a nonexpansive mapping of C into itself. If $\{x_n\}$ is a sequence of C such that $x_n \rightharpoonup x$ and $x_n - Tx_n \to 0$, then x is a fixed point of T.*

Lemma 2.12. (see [42]) *Let E be a Banach space satisfying weakly continuous duality map, K a nonempty closed convex subset of E and let $T : K \longrightarrow K$ be an asymptotically nonexpansive mapping with a fixed point. Then $I - T$ is demiclosed at zero; if $\{x_n\}$ is a sequence of K such that $x_n \rightharpoonup x$ and if $x_n - Tx_n \to 0$, then $x - Tx = 0$.*

Lemma 2.13. (see [6]) *Let C be a nonempty closed convex subset of a real Hilbert space H and $T : C \longrightarrow C$ be a k-strictly pseudocontractive mapping. Define $A = I - T : C \longrightarrow H$. Then, A is $\dfrac{1-k}{2}$-inverse-strongly accretive mapping; that is, for all $x, y \in C$,*

$$\langle x - y, Ax - Ay\rangle \geq \frac{1-k}{2}\|Ax - Ay\|^2.$$

3. Main Results

Lemma 3.1. *Let C be a nonempty closed convex subset of 2-uniformly smooth and uniformly convex Banach space X, which admits weakly sequentially continuous duality mapping. Assume that C is a sunny nonexpansive retract of X and let Q_C be the sunny nonexpansive retraction of X onto C. Let $A, B : C \longrightarrow X$ be ξ-inverse-strongly accretive and η-inverse-strongly accretive mappings, respectively. Let $f : C \longrightarrow C$ be a ρ-strict contraction of C into itself with coefficient $\rho \in (0, 1)$. Let $S, T : C \longrightarrow C$ be two asymptotically nonexpansive self* *mappings on C such that $\mathcal{F} = F(T) \cap F(S) \cap F(G) \neq \emptyset$, where G is as defined by Lemma 2.9. For arbitrarily chosen $x_1 \in C$, let the sequence $\{x_n\}_{n\geq 1}$ be defined iteratively as follows:*

$$\begin{cases} x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n-\gamma_n)x_n + \gamma_n T^n y_n; \\ y_n = \alpha'_n f(x_n) + (1-\alpha'_n-\gamma'_n)x_n + \gamma'_n S^n z_n; \\ z_n = Q_C(I-\lambda A)u_n; \\ u_n = Q_C(I-\mu B)x_n, \end{cases} \tag{3.1}$$

where $0 < \lambda < \frac{\xi}{K^2}, 0 < \mu < \frac{\eta}{K^2}$. Suppose $\{\alpha_n\}, \{\beta_n = 1-\alpha_n-\gamma_n\}, \{\gamma_n\}, \{\alpha'_n\}, \{\beta'_n = 1-\alpha'_n-\gamma'_n\}, \{\gamma'_n\}$ are sequences in $[0,1]$ satisfying the following conditions:

(a) $0 < \alpha'_n \leq \alpha_n \leq \gamma'_n \leq \gamma_n < 1$;
(b) $0 < \liminf \beta_n \leq \limsup \beta_n < 1$;
(c) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty, \sum_{n=1}^{\infty} \gamma_n^2 < \infty, \lim_{n\to\infty} \frac{k_n - 1}{\alpha_n} = 0$;
(d) *T, S satisfy the asymptotically regularity:*
$\lim_{n\to\infty} \|T^{n+1}x_n - T^n x_n\| = 0 = \lim_{n\to\infty} \left\|S^{n+1}x_n - S^n x_n\right\|$.

Then, the sequence $\{x_n\}$ is bounded and $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$.

Proof. Let $k_n = max\{k_n^{(1)}, k_n^{(2)}\}$. Firstly, we prove that $\{x_n\}$ is bounded. Let $x^\star \in \mathcal{F}$. Then, it follows from Lemma 2.9 that $x^\star = Q_C(Q_C(I-\mu B)x^\star - \lambda A Q_C(I-\mu B)x^\star))$. Let $t^\star = Q_C(I-\mu B)x^\star)$, then $x^\star = Q_C(I-\lambda A)t^\star)$. Also, from Lemma 2.9, we have

$$\|z_n - x^\star\| = \|Gx_n - Gx^\star\| \leq \|x_n - x^\star\| \tag{3.2}$$

By condition (c), there exists a constant ϵ with $0 < \epsilon < 1-\delta$ and $\gamma_n(k_n - 1) < \epsilon\alpha_n$ such that following estimates hold:

$$\begin{aligned} \|x_{n+1} - x^\star\| &= \|\alpha_n f(x_n) + (1-\alpha_n-\gamma_n)x_n + \gamma_n T^n y_n - x^\star\| \\ &= \|\alpha_n(f(x_n) - x^\star) + (1-\alpha_n-\gamma_n)(x_n - x^\star) + \gamma_n(T^n y_n - x^\star)\| \\ &\leq \alpha_n\|f(x_n) - x^\star\| + (1-\alpha_n-\gamma_n)\|x_n - x^\star\| + \gamma_n\|T^n y_n - x^\star\| \\ &\leq \alpha_n\rho\|x_n - x^\star\| + \alpha_n\|f(x^\star) - x^\star\| + (1-\alpha_n-\gamma_n)\|x_n - x^\star\| + \gamma_n k_n^{(1)}\|y_n - x^\star\| \\ &\leq (1-(1-\rho)\alpha_n)\|x_n - x^\star\| + \alpha_n\|f(x^\star) - x^\star\| - \gamma_n\|x_n - x^\star\| \\ &\quad + \gamma_n k_n\|y_n - x^\star\| \end{aligned} \tag{3.3}$$

Again, from (3.1), we get

$$\begin{aligned} \|y_n - x^\star\| &= \|\alpha'_n f(x_n) + (1-\alpha'_n-\gamma'_n)x_n + \gamma'_n S^n z_n - x^\star\| \\ &= \|\alpha'_n(f(x_n) - x^\star) + (1-\alpha'_n-\gamma'_n)(x_n - x^\star) + \gamma'_n(S^n z_n - x^\star)\| \\ &\leq \alpha'_n\|f(x_n) - x^\star\| + (1-\alpha'_n-\gamma'_n)\|x_n - x^\star\| + \gamma'_n\|S^n z_n - x^\star\| \\ &\leq \alpha'_n\rho\|x_n - x^\star\| + \alpha'_n\|f(x^\star) - x^\star\| + (1-\alpha'_n-\gamma'_n)\|x_n - x^\star\| + \gamma'_n k_n^{(2)}\|z_n - x^\star\| \\ &\leq (1-(1-\rho)\alpha'_n)\|x_n - x^\star\| + \alpha'_n\|f(x^\star) - x^\star\| + \gamma'_n(k_n - 1)\|x_n - x^\star\| \\ &\leq (1-(1-\rho-\epsilon)\alpha'_n)\|x_n - x^\star\| + \alpha'_n\|f(x^\star) - x^\star\| \end{aligned} \tag{3.4}$$

(3.3) and (3.4) imply

$$\begin{aligned}
\|x_{n+1}-x^\star\| &\le (1-(1-\rho)\alpha_n)\|x_n-x^\star\|+\alpha_n\|f(x^\star)-x^\star\| \\
&\quad -\gamma_n\|x_n-x^\star-(\alpha_n f(x^\star)-x^\star)+(\alpha_n f(x^\star)-x^\star)\| \\
&\quad +\gamma_n k_n\{(1-(1-\rho-\epsilon)\alpha'_n)\|x_n-x^\star\|+\alpha_n\|f(x^\star)-x^\star\|\} \\
&\le (1-(1-\rho)\alpha_n)\|x_n-x^\star\|+\alpha_n\|f(x^\star)-x^\star\| \\
&\quad -\gamma_n\|x_n-x^\star\|-\gamma_n\alpha_n\|f(x^\star)-x^\star\|-\gamma_n\alpha_n\|f(x^\star)-x^\star\| \\
&\quad +\gamma_n k_n\{(1-(1-\rho-\epsilon)\alpha'_n)\|x_n-x^\star\|+\alpha_n\|f(x^\star)-x^\star\|\} \\
&= (1-(1-\rho)\alpha_n)\|x_n-x^\star\|+[1+\gamma_n(k_n-2)]\alpha_n\|f(x^\star)-x^\star\| \\
&\quad +\gamma_n\{k_n[(1-(1-\rho-\epsilon)\alpha'_n)]-1\}\|x_n-x^\star\| \qquad (3.5)
\end{aligned}$$

Using conditions (a) and (c), we get

$$\begin{aligned}
1+\gamma(k_n-2) &= 1+\gamma_n(k_n-1)-\gamma_n \\
&< 1+\epsilon\alpha_n-\gamma_n \\
&< 1
\end{aligned}$$

Applying the above information in (3.5), we obtain

$$\begin{aligned}
\|x_{n+1}-x^\star\| &\le (1-(1-\rho)\alpha_n)\|x_n-x^\star\|+\alpha_n\|f(x^\star)-x^\star\| \\
&\quad +\gamma_n(k_n-1)\|x_n-x^\star\| \\
&< (1-(1-\rho-\epsilon)\alpha_n)\|x_n-x^\star\|+\alpha_n\|f(x^\star)-x^\star\| \\
&= (1-(1-\rho-\epsilon)\alpha_n)\|x_n-x^\star\|+(1-\rho-\epsilon)\alpha_n\frac{\|f(x^\star)-x^\star\|}{(1-\rho-\epsilon)} \\
&\le max\Big\{\|x_n-x^\star\|,\frac{\|f(x^\star)-x^\star\|}{(1-\rho-\epsilon)}\Big\}
\end{aligned}$$

By induction, we have

$$\|x_n-x^\star\|\le max\Big\{\|x_1-x^\star\|,\frac{\|f(x^\star)-x^\star\|}{(1-\rho-\epsilon)}\Big\},\forall n\ge 1,$$

which implies that the sequence $\{x_n\}$ is bounded, and so are the sequences $\{z_n\}$and $\{y_n\}$. Next, we show that $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$. Define the sequence $\{x_n\}$ as follows:

$$x_{n+1}=(1-\alpha_n-\gamma_n)x_n+(\alpha_n+\gamma_n)y_n, \qquad (3.6)$$

where $\alpha_n + \gamma_n \in (0,1)$. Then, it follows that

$$\begin{aligned}
y_{n+1} - y_n &= \frac{x_{n+2} - (1 - \alpha_{n+1} - \gamma_{n+1})x_{n+1}}{\alpha_{n+1} + \gamma_{n+1}} - \frac{x_{n+1} - (1 - \alpha_n - \gamma_n)x_n}{\alpha_n + \gamma_n} \\
&= \frac{\alpha_{n+1} f(x_{n+1}) + \gamma_{n+1} T^{n+1} y_{n+1}}{\alpha_{n+1} + \gamma_{n+1}} - \frac{\alpha_n f(x_n) + \gamma_n T^n y_n}{\alpha_n + \gamma_n} \\
&= \frac{\alpha_{n+1} f(x_{n+1}) - (\alpha_{n+1} + \gamma_{n+1}) T^{n+1} x_{n+1} + \gamma_{n+1} T^{n+1} y_{n+1}}{\alpha_{n+1} + \gamma_{n+1}} \\
&\quad - \frac{\alpha_n f(x_n) - (\alpha_n + \gamma_n) T^n x_n + \gamma_n T y_n}{\alpha_n + \gamma_n} + T^{n+1} x_{n+1} - T^n x_n \\
&= \frac{\alpha_{n+1}(f(x_{n+1}) - T^{n+1} x_{n+1}) + \gamma_{n+1}(T y_{n+1} - T^{n+1} x_{n+1})}{\alpha_{n+1} + \gamma_{n+1}} \\
&\quad - \frac{\alpha_n(f(x_n) - T^n x_n) + \gamma_n(T y_n - T^n x_n)}{\alpha_n + \gamma_n} + T^{n+1} x_{n+1} - T^n x_n \qquad (3.7)
\end{aligned}$$

(3.7) implies that

$$\begin{aligned}
&\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\| \\
\leq\ & \frac{\alpha_{n+1}}{\alpha_{n+1} + \gamma_{n+1}} \|f(x_{n+1}) - T^{n+1} x_{n+1}\| \\
&+ \frac{\gamma_{n+1}}{\alpha_{n+1} + \gamma_{n+1}} \|T^{n+1} y_{n+1} - T^{n+1} x_{n+1}\| + \frac{\alpha_n}{\alpha_n + \gamma_n} \|f(x_n) - T^n x_n\| \\
&+ \frac{\gamma_n}{\alpha_n + \gamma_n} \|T^n y_n - T^n x_n\| + \|T^{n+1} x_{n+1} - T^{n+1} x_n\| \\
&+ \|T^{n+1} x_n - T^n x_n\| - \|x_{n+1} - x_n\| \\
\leq\ & \frac{\alpha_{n+1}}{\alpha_{n+1} + \gamma_{n+1}} \|f(x_{n+1}) - T^{n+1} x_{n+1}\| \\
&+ \frac{\gamma_{n+1} k_{n+1}}{\alpha_{n+1} + \gamma_{n+1}} \|x_{n+1} - y_{n+1}\| + \frac{\alpha_n}{\alpha_n + \gamma_n} \|f(x_n) - T^n x_n\| \\
&+ \frac{\gamma_n k_n}{\alpha_n + \gamma_n} \|x_n - y_n\| + (k_{n+1} - 1)\|x_{n+1} - x_n\| \\
&+ \|T^{n+1} x_n - T^n x_n\| \qquad (3.8)
\end{aligned}$$

Observe that

$$\begin{aligned}
\|x_{n+1} - y_{n+1}\| &= \|x_{n+1} - (\alpha'_{n+1} f(x_{n+1}) + (1 - \alpha'_{n+1} - \gamma'_{n+1}) x_{n+1} + \gamma'_{n+1} S^{n+1} x_{n+1})\| \\
&= \|(\alpha'_{n+1} + \gamma'_{n+1})(x_{n+1} - S^{n+1} x_{n+1}) + \alpha'_{n+1}(S^{n+1} x_{n+1} - f(x_{n+1}))\| \\
&\leq (\alpha'_{n+1} + \gamma'_{n+1}) \|x_{n+1} - S^{n+1} x_{n+1}\| + \alpha'_{n+1} \|S^{n+1} x_{n+1} - f(x_{n+1}\| \\
&\leq (\alpha_{n+1} + \gamma_{n+1}) \|x_{n+1} - S^{n+1} x_{n+1}\| + \alpha_{n+1} \|S^{n+1} x_{n+1} - f(x_{n+1}\| \quad (3.9)
\end{aligned}$$

(3.8) and (3.9) imply that

$$\begin{aligned}\|y_{n+1}-y_n\|-\|x_{n+1}-x_n\| &\leq \frac{\alpha_{n+1}}{\alpha_{n+1}+\gamma_{n+1}}\|f(x_{n+1})-T^{n+1}x_{n+1}\| \\ &+\frac{\gamma_{n+1}k_{n+1}}{\alpha_{n+1}+\gamma_{n+1}}[(\alpha_{n+1}+\gamma_{n+1})\|x_{n+1}-S^{n+1}x_{n+1}\| \\ &+\alpha_{n+1}\|S^{n+1}x_{n+1}-f(x_{n+1}\|]+\frac{\alpha_n}{\alpha_n+\gamma_n}\|f(x_n)-T^nx_n\| \\ &+\frac{\gamma_nk_n}{\alpha_n+\gamma_n}[(\alpha_n+\gamma_n)\|x_n-S^nx_n\|+\alpha_n\|S^nx_n-f(x_n\|] \\ &+(k_{n+1}-1)\|x_{n+1}-x_n\|+\|T^{n+1}x_n-T^nx_n\| \qquad (3.10)\end{aligned}$$

By conditions (a), (c),(d) and the fact that $\lim_{n\to\infty} k_{n+1}=1$, we obtain from (3.10) that

$$\limsup_{n\to\infty}(\|y_{n+1}-y_n\|-\|x_{n+1}-x_n\|)\leq 0,$$

which from Lemma 2.5 yields

$$\lim_{n\to\infty}\|y_n-x_n\|=0 \qquad (3.11)$$

(3.6) together with (3.11) gives

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=\lim_{n\to\infty}(\alpha_n+\gamma_n)\|y_n-x_n\|=0 \qquad (3.12)$$

□

Lemma 3.2. *Let C be a nonempty closed convex subset of 2-uniformly smooth and uniformly convex Banach space X, which admits weakly sequentially continuous duality mapping. Assume that C is a sunny nonexpansive retract of X and let Q_C be the sunny nonexpansive retraction of X onto C. Let $A, B : C \longrightarrow X$ be ξ-inverse-strongly accretive and η-inverse-strongly accretive mappings, respectively. Let $f : C \longrightarrow C$ be a ρ-strict contraction of C into itself with coefficient $\rho\in(0,1)$. Let $S, T : C \longrightarrow C$ be asymptotically nonexpansive self mappings on C such that $\mathcal{F}=F(T)\cap F(S)\cap F(G)\neq\emptyset$, where G is as defined by Lemma 2.9. For arbitrarily chosen $x_1\in C$, let the sequence $\{x_n\}_{n\geq1}$ be defined iteratively as follows:*

$$\begin{cases} x_{n+1}=\alpha_nf(x_n)+(1-\alpha_n-\gamma_n)x_n+\gamma_nT^ny_n;\\ y_n=\alpha'_nf(x_n)+(1-\alpha'_n-\gamma'_n)x_n+\gamma'_nS^nz_n;\\ z_n=Q_C(I-\lambda A)u_n;\\ u_n=Q_C(I-\mu B)x_n,\end{cases} \qquad (3.13)$$

where $0<\lambda<\frac{\xi}{K^2}, 0<\mu<\frac{\eta}{K^2}$. Suppose $\{\alpha_n\},\{\beta_n\},\{\gamma_n\},\{\alpha'_n\},\{\beta'_n\},\{\gamma'_n\}$ are sequences in $[0,1]$ satisfying conditons of Lemma 3.1. Then, $\lim_{n\to\infty}\|x_n-z_n\|=0, \lim_{n\to\infty}\|x_n-Sx_n\|=0$ and $\lim_{n\to\infty}\|x_n-Tx_n\|=0$.

Proof. From

$$\begin{aligned}\|x_{n+1}-T^ny_n\| &= \|\alpha_nf(x_n)+\beta_nx_n+\gamma_nT^ny_n-T^ny_n\| \\ &= \|\alpha_n(f(x_n)-T^ny_n)+\beta_n(x_n-T^ny_n)\| \\ &\leq \alpha_n\|f(x_n)-T^ny_n\|+\beta_n\|x_n-T^ny_n\| \\ &\leq \alpha_n\|f(x_n)-T^ny_n\|+\beta_n\|x_n-x_{n+1}\|+\beta_n\|x_{n+1}-T^ny_n\|,\end{aligned}$$

we get

$$\|x_{n+1}-T^ny_n\| \leq \frac{\alpha_n}{1-\beta_n}\|f(x_n)-T^ny_n\| + \frac{\beta_n}{1-\beta_n}\|x_n-x_{n+1}\| \to 0 \quad \text{as} \quad n\to\infty. \tag{3.14}$$

Since

$$\begin{aligned}
& \|x_n - Ty_n| \\
\leq\ & \|x_n-x_{n+1}\| + \|x_{n+1}-T^ny_n\| + \|T^ny_n - T^ny_{n-1}\| + \|T^ny_{n-1}-Ty_n\| \\
\leq\ & \|x_n-x_{n+1}\| + \|x_{n+1}-T^ny_n\| + k_n^{(1)}\|y_n-y_{n-1}\| + \|T(T^{n-1}y_{n-1})-Ty_n\| \\
\leq\ & \|x_n-x_{n+1}\| + \|x_{n+1}-T^ny_n\| + k_n^{(1)}\|y_n-y_{n-1}\| + k_n^{(1)}\|T^{n-1}y_{n-1}-y_n\| \\
\leq\ & \|x_n-x_{n+1}\| + \|x_{n+1}-T^ny_n\| + 2k_n\|y_n-x_n\| + k_n|x_n-x_{n-1}\| + k_n\|x_{n-1}-y_{n-1}\| \\
& +k_n\|T^{n-1}y_{n-1}-x_n\|,
\end{aligned}$$

it follows from (3.11), (3.12) and (3.14) that

$$\lim_{n\to\infty}\|x_n-Ty_n|=0. \tag{3.15}$$

Observe that

$$\begin{aligned}
\|x_n-Tx_n\| &\leq \|x_n-Ty_n|+\|Tx_n-Ty_n| \\
&\leq \|x_n-Ty_n|+k_n\|x_n-y_n|\to 0 \quad \text{as} \quad n\to\infty.
\end{aligned}\tag{3.16}$$

Furthermore, from (3.13) and Lemma 2.8, we obtain

$$\begin{aligned}
\|u_n-t^\star\|^2 &= \|Q_C(I-\mu B)x_n) - Q_C(I-\mu B)x^\star)\|^2 \\
&\leq \|(I-\mu B)x_n - (I-\mu B)x^\star\|^2 \\
&= \|x_n - x^\star - \mu(Bx_n - Bx^\star)\|^2 \\
&\leq \|x_n-x^\star\|^2 - 2\mu(\xi-K^2\xi)\|Bx_n-Bx^\star\|^2
\end{aligned}\tag{3.17}$$

and

$$\begin{aligned}
\|z_n-x^\star\|^2 &= \|Q_C(I-\lambda A)u_n) - Q_C(I-\lambda A)t^\star\|^2) \\
&\leq \|(I-\lambda A)u_n - (I-\lambda A)t^\star\|^2 \\
&= \|u_n-t^\star-\lambda(Au_n-At^\star)\|^2 \\
&\leq \|u_n-t^\star\|^2 - 2\lambda(\eta-K^2\eta)\|Au_n-At^\star\|^2
\end{aligned}\tag{3.18}$$

(3.17) and (3.18) imply

$$\|z_n-x^\star\|^2 \leq \|x_n-x^\star\|^2 - 2\mu(\xi-K^2\xi)\|Bx_n-Bx^\star\|^2 - 2\lambda(\eta-K^2\eta)\|Au_n-At^\star\|^2 \tag{3.19}$$

Again, from (3.13) and the convexity of $\|,.,\|^2$, we get

$$\begin{aligned}
\|x_{n+1}-x^\star\|^2 &= \|\alpha_nf(x_n)+(1-\alpha_n-\gamma_n)x_n+\gamma_nT^ny_n-x^\star\|^2 \\
&= \|\alpha_n(f(x_n)-x^\star)+(1-\alpha_n-\gamma_n)(x_n-x^\star)+\gamma_n(T^ny_n-x^\star)\|^2 \\
&\leq \alpha_n\|f(x_n)-x^\star\|^2+(1-\alpha_n-\gamma_n)\|x_n-x^\star\|^2+\gamma_n\|T^ny_n-x^\star\|^2 \\
&\leq \alpha_nM^\star+(1-\alpha_n-\gamma_n)\|x_n-x^\star\|^2+\gamma_n(k_n^{(1)})^2\|y_n-x^\star\|^2,
\end{aligned}\tag{3.20}$$

where $M^\star = \sup_{n\geq 1} = \|f(x_n)-x^\star\|^2$.

Similarly, from (3.13) and the convexity of $\|.,.\|^2$, with $\alpha_n, \gamma_n, x_{n+1}$ and $T^n y_n$ replaced by $\alpha'_n, \gamma'_n, y_n$ and $S^n y_n$ in (3.20), respectively, we have

$$\|y_n - x^\star\|^2 \leq \alpha'_n M^\star + (1-\alpha'_n-\gamma'_n)\|x_n - x^\star\|^2 + \gamma_n (k_n^{(2)})^2 \|z_n - x^\star\|^2, \quad (3.21)$$

From (3.19), (3.20) and (3.21), we obtain

$$\begin{aligned}
&\|x_{n+1} - x^\star\|^2 \\
\leq\; & \alpha_n M^\star + (1-\alpha_n-\gamma_n)\|x_n - x^\star\|^2 + \gamma_n k_n^2\{\alpha'_n M^\star + (1-\alpha'_n-\gamma'_n)\|x_n - x^\star\|^2 \\
& +\gamma'_n k_n^2[\|x_n - x^\star\|^2 - 2\mu(\xi - K^2\xi)\|Bx_n - Bx^\star\|^2 - 2\lambda(\eta - K^2\eta)\|Au_n - At^\star\|^2]\} \\
=\; & \alpha_n M^\star + (1-\alpha_n-\gamma_n)\|x_n - x^\star\|^2 + \gamma_n k_n^2\{\alpha'_n M^\star + (1-\alpha'_n)\|x_n - x^\star\|^2 \\
& +\gamma'_n(k_n^2 - 1)\|x_n - x^\star\|^2 - 2\gamma'_n k_n^2 \mu(\xi - K^2\xi)\|Bx_n - Bx^\star\|^2 \\
& -2\gamma'_n k_n^2 \lambda(\eta - K^2\eta)\|Au_n - At^\star\|^2\} \\
\leq\; & \alpha_n M^\star + (1-\alpha_n-\gamma_n)\|x_n - x^\star\|^2 + \alpha_n\gamma_n k_n^2 M^\star + \gamma_n k_n^2(1-\alpha'_n)\|x_n - x^\star\|^2 \\
& +\gamma_n^2 k_n^2(k_n^2-1)\|x_n - x^\star\|^2 - 2\gamma'_n\gamma_n k_n^4\mu(\xi - K^2\xi)\|Bx_n - Bx^\star\|^2 \\
& -2\gamma'_n\gamma_n k_n^4\lambda(\eta - K^2\eta)\|Au_n - At^\star\|^2 \\
\leq\; & \alpha_n(1+\gamma_n k_n^2)M^\star + (1-\alpha_n)\|x_n - x^\star\|^2 + \gamma_n(k_n^2 - 1)\|x_n - x^\star\|^2 \\
& +\gamma_n^2 k_n^2(k_n^2-1)\|x_n - x^\star\|^2 - 2\gamma'_n\gamma_n k_n^4\mu(\xi - K^2\xi)\|Bx_n - Bx^\star\|^2 \\
& -2\gamma'_n\gamma_n k_n^4\lambda(\eta - K^2\eta)\|Au_n - At^\star\|^2
\end{aligned}$$

Set $W = 2\gamma'_n\gamma_n k_n^4\mu(\xi - K^2\xi)\|Bx_n - Bx^\star\|^2 + 2\gamma'_n\gamma_n k_n^4\lambda(\eta - K^2\eta)\|Au_n - At^\star\|^2$. Then, it follows from the last inequality that

$$\begin{aligned}
W \;\leq\; & \alpha_n(1+\gamma_n k_n^2)M^\star + (1-\alpha_n)\|x_n - x^\star\|^2 - \|x_{n+1} - x^\star\|^2 + \gamma_n(1+\gamma_n k_n^2)(k_n^2-1)\|x_n - x^\star\|^2 \\
=\; & \alpha_n(1+\gamma_n k_n^2)M^\star + \|-(x_{n+1} - x_n) + x_{n+1} - x^\star\|^2 - \|x_{n+1} - x^\star\|^2 \\
& +\gamma_n(1+\gamma_n k_n^2)(k_n^2-1)\|x_n - x^\star\|^2 \\
\leq\; & \alpha_n(1+\gamma_n k_n^2)M^\star + \|x_{n+1} - x_n)\|^2 + \gamma_n(1+\gamma_n k_n^2)(k_n^2-1)\|x_n - x^\star\|^2 \qquad (3.22)
\end{aligned}$$

Using (3.12), (3.22), conditions (b), (c), Lemma 3.1 and the fact that $0 < \lambda < \dfrac{\xi}{K^2}, 0 < \mu < \dfrac{\eta}{K^2}$ and $\lim_{n\longrightarrow\infty} k_n = 1$, we get

$$\lim_{n\longrightarrow\infty} W = 0$$

Thus,

$$\lim_{n\longrightarrow\infty}\|Bx_n - Bx^\star\| = 0 = \lim_{n\longrightarrow\infty}\|Au_n - At^\star\| \quad (3.23)$$

Since, using Lemma 2.7 and Lemma 2.2,

$$\begin{aligned}
\|u_n - t^\star\|^2 \;=\; & \|Q_C(I-\mu B)x_n) - Q_C(I-\mu B)x^\star)\|^2 \\
\leq\; & \langle x_n - \mu Bx_n - (x^\star - \mu Bx^\star), j(u_n - t^\star)\rangle \\
=\; & \langle x_n - x^\star, j(u_n - t^\star)\rangle + \mu\langle Bx^\star - Bx_n, j(u_n - t^\star)\rangle \\
\leq\; & \frac{1}{2}[\|x_n - x^\star\|^2 + \|u_n - t^\star\|^2 - g^\star(\|x_n - u_n - (x^\star - t^\star)\|)] + \mu\|Bx_n - Bx^\star\| \\
& \times\|u_n - t^\star\|,
\end{aligned}$$

it follows that

$$\|u_n - t^\star\|^2 \leq \|x_n - x^\star\|^2 - g^\star(\|x_n - u_n - (x^\star - t^\star)\|) + 2\mu\|Bx_n - Bx^\star\|\|u_n - t^\star\| \quad (3.24)$$

Similarly, since

$$\begin{aligned} \|z_n - x^\star\|^2 &= \|Q_C(I-\lambda A)u_n) - Q_C(I-\lambda A)t^\star)\|^2 \\ &\leq \langle u_n - \lambda A u_n - (t^\star - \lambda A t^\star), j(z_n - x^\star\rangle \\ &= \langle u_n - t^\star, j(z_n - x^\star\rangle + \lambda\langle At^\star - Au_n, j(z_n - x^\star\rangle \\ &\leq \frac{1}{2}[\|u_n - t^\star\|^2 + \|z_n - x^\star\|^2 - g^{\star\star}(\|u_n - z_n + (x^\star - t^\star)\|)] + \lambda\|At^\star - Au_n\| \\ &\quad \times\|z_n - x^\star\|, \end{aligned}$$

it follows that

$$\|z_n - x^\star\|^2 \leq \|u_n - t^\star\|^2 - g^{\star\star}(\|u_n - z_n + (x^\star - t^\star)\|) + 2\lambda\|At^\star - Au_n\|\|z_n - x^\star\| \quad (3.25)$$

(3.24) and (3.25) imply that

$$\begin{aligned} \|z_n - x^\star\|^2 &\leq \|x_n - x^\star\|^2 - g^\star(\|x_n - u_n - (x^\star - t^\star)\|) + 2\mu\|Bx_n - Bx^\star\|\|u_n - t^\star\| \\ &\quad -g^{\star\star}(\|u_n - z_n + (x^\star - t^\star)\|) + 2\lambda\|At^\star - Au_n\|\|z_n - x^\star\| \end{aligned} \quad (3.26)$$

From (3.21) and (3.26), we have

$$\begin{aligned} \|y_n - x^\star\|^2 &\leq \alpha_n' M^\star + (1-\alpha_n' - \gamma_n')\|x_n - x^\star\|^2 + \gamma_n' k_n^2[\|x_n - x^\star\|^2 \\ &\quad -g^\star(\|x_n - u_n - (x^\star - t^\star)\|) + 2\mu\|Bx_n - Bx^\star\|\|u_n - t^\star\| \\ &\quad -g^{\star\star}(\|u_n - z_n + (x^\star - t^\star)\|) + 2\lambda\|At^\star - Au_n\|\|z_n - x^\star\|] \\ &= \alpha_n' M^\star + (1-\alpha_n')\|x_n - x^\star\|^2 + \gamma_n'(k_n^2 - 1)\|x_n - x^\star\|^2 \\ &\quad -\gamma_n' k_n^2 g^\star(\|x_n - u_n - (x^\star - t^\star)\|) + 2\mu\gamma_n' k_n^2\|Bx_n - Bx^\star\|\|u_n - t^\star\| \\ &\quad -\gamma_n' k_n^2 g^{\star\star}(\|u_n - z_n + (x^\star - t^\star)\|) + 2\lambda\gamma_n' k_n^2\|At^\star - Au_n\| \\ &\quad \times\|z_n - x^\star\| \end{aligned} \quad (3.27)$$

Thus, from (3.20), (3.27) and the inequality:

$$\begin{aligned} \|x_{n+1} - x^\star\|^2 &\leq \alpha_n M^\star + (1-\alpha_n-\gamma_n)\|x_n - x^\star\|^2 + \gamma_n k_n^2[\alpha_n' M^\star + (1-\alpha_n')\|x_n - x^\star\|^2 \\ &\quad +\gamma_n'(k_n^2-1)\|x_n - x^\star\|^2 - \gamma_n' k_n^2 g^\star(\|x_n - u_n - (x^\star - t^\star)\|) \\ &\quad +2\mu\gamma_n' k_n^2\|Bx_n - Bx^\star\|\|u_n - t^\star\| - \gamma_n' k_n^2 g^{\star\star}(\|u_n - z_n + (x^\star - t^\star)\|) \\ &\quad +2\lambda\gamma_n' k_n^2\|At^\star - Au_n\|\|z_n - x^\star\|] \\ &\leq \alpha_n M^\star(1+\gamma_n k_n^2) + (1-\alpha_n)\|x_n - x^\star\|^2 + \gamma_n(k_n^2-1)\|x_n - x^\star\|^2 \\ &\quad +\gamma_n'\gamma_n k_n^2(k_n^2-1)\|x_n - x^\star\|^2 - \gamma_n'\gamma_n k_n^4 g^\star(\|x_n - u_n - (x^\star - t^\star)\|) \\ &\quad +2\mu\gamma_n'\gamma_n k_n^4\|Bx_n - Bx^\star\|\|u_n - t^\star\| - \gamma_n'\gamma_n k_n^4 g^{\star\star}(\|u_n - z_n + (x^\star - t^\star)\|) \\ &\quad +2\lambda\gamma_n'\gamma_n k_n^4\|At^\star - Au_n\|\|z_n - x^\star\| \\ &= \alpha_n M^\star(1+\gamma_n k_n^2) + (1-\alpha_n)\|x_n - x^\star\|^2 + \gamma_n(1+\gamma_n' k_n^2)(k_n^2-1)\|x_n - x^\star\|^2 \\ &\quad -\gamma_n'\gamma_n k_n^4 g^\star(\|x_n - u_n - (x^\star - t^\star)\|) + 2\mu\gamma_n'\gamma_n k_n^4\|Bx_n - Bx^\star\|\|u_n - t^\star\| \\ &\quad -\gamma_n'\gamma_n k_n^4 g^{\star\star}(\|u_n - z_n + (x^\star - t^\star)\|) + 2\lambda\gamma_n'\gamma_n k_n^4\|At^\star - Au_n\|\|z_n - x^\star\|, \end{aligned}$$

we obtain, with $W^\star = \gamma_n'\gamma_n k_n^4 g^\star(\|x_n - u_n - (x^\star - t^\star)\|) + \gamma_n'\gamma_n k_n^4 g^{\star\star}(\|u_n - z_n + (x^\star - t^\star)\|)$, the following estimation:

$$\begin{aligned} W^\star &\leq \alpha_n M^\star(1+\gamma_n k_n^2) + (1-\alpha_n)\|x_n - x^\star\|^2 - \|x_{n+1} - x^\star\|^2 \\ &\quad +\gamma_n(1+\gamma_n' k_n^2)(k_n^2 - 1)\|x_n - x^\star\|^2 + 2\mu\gamma_n'\gamma_n k_n^4\|Bx_n - Bx^\star\|\|u_n - t^\star\| \\ &\quad +2\lambda\gamma_n'\gamma_n k_n^4\|At^\star - Au_n\|\|z_n - x^\star\| \\ &\leq \alpha_n M^\star(1+\gamma_n k_n^2) + \|-(x_{n+1} - x_n) + x_{n+1} - x^\star\|^2 - \|x_{n+1} - x^\star\|^2 \\ &\quad +\gamma_n(1+\gamma_n' k_n^2)(k_n^2 - 1)\|x_n - x^\star\|^2 + 2\mu\gamma_n'\gamma_n k_n^4\|Bx_n - Bx^\star\|\|u_n - t^\star\| \\ &\quad +2\lambda\gamma_n'\gamma_n k_n^4\|At^\star - Au_n\|\|z_n - x^\star\| \\ &\leq \alpha_n M^\star(1+\gamma_n k_n^2) + \|x_{n+1} - x_n\| + \|x_{n+1} - x^\star\|^2 - \|x_{n+1} - x^\star\|^2 \\ &\quad +\gamma_n(1+\gamma_n' k_n^2)(k_n^2 - 1)\|x_n - x^\star\|^2 + 2\mu\gamma_n'\gamma_n k_n^4\|Bx_n - Bx^\star\|\|u_n - t^\star\| \\ &\quad +2\lambda\gamma_n'\gamma_n k_n^4\|At^\star - Au_n\|\|z_n - x^\star\| \end{aligned}$$

Using conditions (b), (c), Lemma 3.1 and the fact that $0 < \lambda < \dfrac{\xi}{K^2}$ and $0 < \mu < \dfrac{\eta}{K^2}$ and $\lim_{n\to\infty} k_n = 1$, we get from the last inequality that

$$\lim_{n\to\infty} \gamma_n'\gamma_n k_n^4 g^\star(\|x_n - u_n - (x^\star - t^\star)\|) = 0 = \lim_{n\to\infty} \gamma_n'\gamma_n k_n^4 g^{\star\star}(\|u_n - z_n + (x^\star - t^\star)\|) \quad (3.28)$$

Thus, from the properties of $g^\star$ and $g^{\star\star}$, we get

$$\lim_{n\to\infty} \|x_n - u_n - (x^\star - t^\star)\| = 0 = \lim_{n\to\infty} \|u_n - z_n + (x^\star - t^\star)\| \quad (3.29)$$

Using (3.29) and the inequality

$$\|x_n - z_n\| \leq \|x_n - u_n - (x^\star - t^\star)\| + \|u_n - z_n + (x^\star - t^\star)\|, \quad (3.30)$$

we obtain

$$\lim_{n\to\infty} \|x_n - z_n\| = 0. \quad (3.31)$$

Since

$$\begin{aligned} \|y_n - S^n z_n\| &= \|\alpha_n' f(x_n) + (1-\alpha_n' - \gamma_n')x_n + \gamma_n' S^n z_n - S^n z_n\| \\ &= \|\alpha_n'(f(x_n) - x_n) + (1-\gamma_n')(x_n - y_n) + (1-\gamma_n')(y_n - S^n z_n)\| \\ &\leq \alpha_n'\|f(x_n) - x_n\| + (1-\gamma_n')\|x_n - y_n\| + (1-\gamma_n')\|y_n - S^n z_n)\|, \end{aligned}$$

it follows from condition (c) and (3.11) that

$$\|y_n - S^n z_n\| \leq \frac{\alpha_n}{\gamma_n'}\|f(x_n) - x_n\| + \frac{(1-\gamma_n')}{\gamma_n'}\|x_n - y_n\| \to 0 \quad \text{as} \quad n \to \infty \quad (3.32)$$

Again, since

$$\|z_n - S^n z_n\| \leq \|z_n - x_n\| + \|x_n - y_n\| + \|y_n - S^n z_n\|$$

and

$$\begin{aligned} \|x_n - S^n x_n\| &\leq \|x_n - z_n\| + \|z_n - S^n z_n\| + \|S^n z_n - S^n x_n\| \\ &\leq (1+k_n)\|x_n - z_n\| + \|z_n - S^n z_n\|, \end{aligned} \quad (3.33)$$

it follows from (3.11), (3.31), (3.32) and (3.33) that

$$\lim_{n\to\infty} \|z_n - S^n z_n\| = 0. \quad (3.34)$$

and

$$\lim_{n\to\infty} \|x_n - S^n x_n\| = 0. \tag{3.35}$$

From (3.35) and condition (d), we have

$$\begin{aligned} \|x_n - Sx_n\| &\leq \|x_n - S^n x_n\| + \|S^n x_n - S^{n+1}x_n\| + \|S^{n+1}x_n - S^n x_n\| \\ &\leq (1+k_n)\|x_n - S^n x_n\| + \|S^n x_n - S^{n+1}x_n\| \to 0 \quad \text{as} \quad n\to\infty. \end{aligned} \tag{3.36}$$

□

Theorem 3.1. *Let C be a nonempty closed convex subset of 2-uniformly smooth and uniformly convex Banach space X, which admits weakly sequentially continuous duality mapping. Assume that C is a sunny nonexpansive retract of X and let Q_C be the sunny nonexpansive retraction of X onto C. Let $A, B : C \longrightarrow X$ be ξ-inverse-strongly accretive and η-inverse-strongly accretive mappings, respectively. Let $f : C \longrightarrow C$ be a ρ-strict contraction of C into itself with coefficient $\rho \in (0,1)$. Let $S, T : C \longrightarrow C$ be asymptotically nonexpansive self mappings on C such that $\mathcal{F} = F(T)\cap F(S)\cap F(G) \neq \emptyset$, where G is as defined by Lemma 2.9. For arbitrarily chosen $x_1 \in C$, let the sequence $\{x_n\}_{n\geq 1}$ be defined iteratively as follows:*

$$\begin{cases} x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n-\gamma_n)x_n + \gamma_n T^n y_n; \\ y_n = \alpha_n' f(x_n) + (1-\alpha_n'-\gamma_n')x_n + \gamma_n' S^n z_n; \\ z_n = Q_C(I-\lambda A)u_n; \\ u_n = Q_C(I-\mu B)x_n, \end{cases} \tag{3.37}$$

where $0<\lambda<\dfrac{\xi}{K^2}, 0<\mu<\dfrac{\eta}{K^2}$. Suppose $\{\alpha_n\},\{\beta_n\},\{\gamma_n\},\{\alpha_n'\},\{\beta_n'\},\{\gamma_n'\}$ are sequences in $[0,1]$ satisfying conditions of Lemma 3.1. Then, the sequence $\{x_n\}$ converges strongly to $q^\star = Q_F f(q)$ and $(q, q^\star)$ is a solution of problem 1.18, where $q^\star = Q_C(q-\mu Sq)$ Q_F is the sunny nonexpansive retraction of C onto F.

Proof. Since for all $x, y \in C$,

$$\|Q_F f(x) - Q_F f(y)\| \leq \|f(x)-f(y)\| \leq \|x-y\|,$$

it follows from Banach contraction principle that there exists a unique $q \in C$ such that $Q_F f(q) = q$. By the definition of sunny nonexpansive retraction Q_F, we have $q \in \mathcal{F}$.

Now, we show that

$$\limsup_{n\to} \langle f(q)-q, j(x_n-q)\rangle \leq 0, \tag{3.38}$$

where $Q_F f(q) = q$. Boundedness of $\{x_n\}$ guarantees the existence of a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup z$ as $k\to\infty$ and

$$\limsup_{n\to\infty} \langle f(q)-q, j(x_n-q)\rangle = \limsup_{k\to\infty} \langle f(q)-q, j(x_{n_k}-q)\rangle \tag{3.39}$$

(3.31) and Lemma 2.12 imply that $z \in F(G)$. Again, (3.16), (3.36), and Lemma 2.11 imply that $z \in F(T)\cap F(S)\cap F(G) = \mathcal{F}$.

Since j is weakly sequentially continuous, Lemma 2.2 and (3.39) imply that

$$\begin{aligned}\limsup_{n\to\infty}\langle f(q)-q, j(x_n-q)\rangle &= \limsup_{k\to\infty}\langle f(q)-q, j(x_{n_k}-q)\rangle \\ &= \langle f(q)-q, j(z-q)\rangle \\ &\leq 0,\end{aligned}$$

which is as required by (3.38).
Next, we show that $x_n \to q = Q_F f(q)$ as $n\to\infty$. Now, using (3.1) and Lemma 2.11, we have

$$\begin{aligned}\|x_{n+1}-q\|^2 &= \|(1-\alpha_n-\gamma_n)(x_n-q)+\gamma_n(T^n y_n-q)+\alpha_n(f(x_n)-q)\|^2 \\ &\leq \|(1-\alpha_n-\gamma_n)(x_n-q)+\gamma_n(T^n y_n-q)\|^2+2\alpha_n\langle f(x_n)-q, j(x_{n+1}-q\rangle \\ &\leq ((1-\alpha_n)\|x_n-q\|-\gamma_n\|x_n-q\|+\gamma_n k_n\|y_n-q\|)^2 \\ &\quad +2\alpha_n\rho\|x_n-q\|\|x_{n+1}-q\|+2\alpha_n\langle f(q)-q, j(x_{n+1}-q\rangle \qquad (3.40)\end{aligned}$$

Using condition (a) and the same argument as in (3.4), with $x^\star$ replaced by q, we obtain

$$\begin{aligned}\|y_n-q\| &\leq \alpha_n\rho\|x_n-q\|+\alpha_n\|f(q)-q\|+(1-\alpha_n')\|x_n-q\| \\ &\quad +\gamma_n(k_n-1)\|x_n-q\| \qquad (3.41)\end{aligned}$$

(3.40) and (3.41) imply that

$$\begin{aligned}&\|x_{n+1}-q\|^2 \\ \leq\ &\{(1-\alpha_n)\|x_n-q\|-\gamma_n\|x_n-q\|+\gamma_n k_n[\alpha_n\rho\|x_n-q\| \\ &+\alpha_n\|f(q)-q\|+(1-\alpha_n')\|x_n-q\|+\gamma_n(k_n-1)\|x_n-q\|]\}^2 \\ &+2\alpha_n\rho\|x_n-q\|\|x_{n+1}-q\|+2\alpha_n\langle f(q)-q, j(x_{n+1}-q\rangle\end{aligned}$$

$$
\begin{aligned}
&\leq \{(1-\alpha_n)\|x_n-q\| - \gamma_n\|x_n-q\| + \gamma_n k_n\alpha_n\rho\|x_n-q\| \\
&\quad +\alpha_n\gamma_n k_n\|f(q)-q\| + \gamma_n k_n\|x_n-q\| + \gamma_n^2 k_n(k_n-1)\|x_n-q\|\}^2 \\
&\quad +2\alpha_n\rho\|x_n-q\|\|x_{n+1}-q\| + 2\alpha_n\langle f(q)-q, j(x_{n+1}-q\rangle \\
&\leq \{[(1-\alpha_n)+\gamma_n(k_n-1)+\gamma_n k_n\alpha_n\rho+\gamma_n^2 k_n(k_n-1)]\|x_n-q\| \\
&\quad +\alpha_n\gamma_n k_n\|f(q)-q\|\}^2 + 2\alpha_n\rho\|x_n-q\|\|x_{n+1}-q\| \\
&\quad +2\alpha_n\langle f(q)-q, j(x_{n+1}-q\rangle \\
&\leq \{[(1-\alpha_n)+\gamma_n(k_n-1)(1+\gamma_n k_n)+\gamma_n k_n\alpha_n\rho]\|x_n-q\| \\
&\quad +\alpha_n\gamma_n k_n\|f(q)-q\|\}^2 + \alpha_n\rho(\|x_n-q\|^2+\|x_{n+1}-q\|^2) \\
&\quad +2\alpha_n\langle f(q)-q, j(x_{n+1}-q\rangle \\
&= [(1-\alpha_n)+\gamma_n((k_n-1)(1+\gamma_n k_n)+k_n\alpha_n\rho)]^2\|x_n-q\|^2 \\
&\quad +2\alpha_n\gamma_n k_n[(1-\alpha_n)+\gamma_n(k_n-1)(1+\gamma_n k_n)+\gamma_n k_n\alpha_n\rho]\|x_n-q\|\|f(q)-q\| \\
&\quad +\alpha_n^2\gamma_n^2 k_n^2\|f(q)-q\|^2 + \alpha_n\rho(\|x_n-q\|^2+\|x_{n+1}-q\|^2) \\
&\quad +2\alpha_n\langle f(q)-q, j(x_{n+1}-q\rangle \\
&\leq [1-2\alpha_n+\alpha_n^2+2\gamma_n((k_n-1)(1+\gamma_n k_n)+k_n\alpha_n\rho) \\
&\quad +\gamma_n^2((k_n-1)(1+\gamma_n k_n)+k_n\alpha_n\rho)^2]\|x_n-q\|^2 + \gamma_n^2[(1-\alpha_n) \\
&\quad +\gamma_n(k_n-1)(1+\gamma_n k_n)+\gamma_n k_n\alpha_n\rho]^2\|x_n-q\|^2 + \alpha_n^2 k_n^2\|f(q)-q\|^2 \\
&\quad +\alpha_n^2\gamma_n^2 k_n^2\|f(q)-q\|^2 + \alpha_n\rho(\|x_n-q\|^2+\|x_{n+1}-q\|^2) \\
&\quad +2\alpha_n\langle f(q)-q, j(x_{n+1}-q\rangle \\
&= [1-(2-\rho)\alpha_n]\|x_n-q\|^2 + [\alpha_n^2+2\gamma_n((k_n-1)(1+\gamma_n k_n)+k_n\alpha_n\rho) \\
&\quad +\gamma_n^2((k_n-1)(1+\gamma_n k_n)+k_n\alpha_n\rho)^2]\|x_n-q\|^2 + \gamma_n^2[(1-\alpha_n) \\
&\quad +\gamma_n(k_n-1)(1+\gamma_n k_n)+\gamma_n k_n\alpha_n\rho]^2\|x_n-q\|^2 \\
&\quad +\alpha_n^2 k_n^2\|f(q)-q\|^2 + \alpha_n^2\gamma_n^2 k_n^2\|f(q)-q\|^2 + \alpha_n\rho\|x_{n+1}-q\|^2 \\
&\quad +2\alpha_n\langle f(q)-q, j(x_{n+1}-q\rangle \\
&= [1-(2-\rho)\alpha_n]\|x_n-q\|^2 + [\alpha_n^2+2\gamma_n((k_n-1)(1+\gamma_n k_n)+k_n\alpha_n\rho) \\
&\quad +2\gamma_n^2((k_n-1)(1+\gamma_n k_n)+k_n\alpha_n\rho)^2+\gamma_n^2(1-\alpha_n)]M + \alpha_n^2 k_n^2(1+\gamma_n^2)\|f(q)-q\|^2 \\
&\quad +\alpha_n\rho\|x_{n+1}-q\|^2 + 2\alpha_n\langle f(q)-q, j(x_{n+1}-q\rangle
\end{aligned}
$$

from which we obtain

$$
\begin{aligned}
&\|x_{n+1}-q\|^2 \\
&\leq [1-\frac{2(1-\rho)\alpha_n}{1-\alpha_n\rho}]\|x_n-q\|^2 + \frac{\alpha_n}{1-\alpha_n\rho}\Big\{\alpha_n + \frac{1}{\alpha_n}[2\gamma_n((k_n-1)(1+\gamma_n k_n)+k_n\alpha_n\rho) \\
&\quad +2\gamma_n^2((k_n-1)(1+\gamma_n k_n)+k_n\alpha_n\rho)^2+\gamma_n^2(1-\alpha_n)]M + \alpha_n k_n^2(1+\gamma_n^2)\|f(q)-q\|^2 \\
&\quad +2\langle f(q)-q, j(x_{n+1}-q\rangle\Big\},
\end{aligned}
$$

where $M = \sup_{n\geq 1}\|x_n-q\|^2$.

Put

$$a_n = \frac{2(1-\rho)\alpha_n}{1-\alpha_n\rho}$$

and

$$\begin{aligned} b_n &= \frac{\alpha_n}{1-\alpha_n\rho}\Big\{\alpha_n + \frac{1}{\alpha_n}[2\gamma_n((k_n-1)(1+\gamma_n k_n) + k_n\alpha_n\rho) + 2\gamma_n^2((k_n-1)(1+\gamma_n k_n) + k_n\alpha_n\rho)^2 \\ &\quad + \gamma_n^2(1-\alpha_n)]M + \alpha_n k_n^2(1+\gamma_n^2)\|f(q)-q\|^2 + 2\langle f(q)-q, j(x_{n+1}-q\rangle\Big\}, \end{aligned}$$

so that

$$\begin{aligned} c_n &= \frac{b_n}{a_n} \\ &= \frac{1}{2(1-\rho}\Big)\Big\{\alpha_n + \frac{1}{\alpha_n}[2\gamma_n((k_n-1)(1+\gamma_n k_n) + k_n\alpha_n\rho) + 2\gamma_n^2((k_n-1)(1+\gamma_n k_n) + k_n\alpha_n\rho)^2 \\ &\quad + \gamma_n^2(1-\alpha_n)]M + \alpha_n k_n^2(1+\gamma_n^2)\|f(q)-q\|^2 + 2\langle f(q)-q, j(x_{n+1}-q\rangle\Big\}, \end{aligned}$$

Then, from condition (c) and (3.38), we get

$$a_n \to 0 \quad \text{as} n \to \infty \quad , \sum_{n=1}^{\infty} a_n = \infty \quad \text{and} \quad \limsup_{n\to\infty} c_n \le 0.$$

Thus, from Lemma 2.6, the result follows as required (i.e., $x_n \to q$ as $n \to \infty$) and this completes the proof. □

Corollary 3.1. *Let C be a nonempty closed convex subset of a Hilbert space H. Let P_C be a metric projection from H onto C. Let $A, B : C \longrightarrow H$ be ξ-inverse-strongly accretive and η-inverse-strongly accretive mappings, respectively. Let $f : C \longrightarrow C$ be a ρ-strict contraction of C into itself with coefficient $\rho \in (0,1)$. Let $S, T : C \longrightarrow C$ be asymptotically nonexpansive self mappings on C such that $\mathcal{F} = F(T) \cap F(S) \cap F(G) \neq \emptyset$, where G is as defined by Lemma 2.9. For arbitrarily chosen $x_1 \in C$, let the sequence $\{x_n\}_{n\ge1}$ be defined iteratively as follows:*

$$\begin{cases} x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n-\gamma_n)x_n + \gamma_n T^n y_n; \\ y_n = \alpha'_n f(x_n) + (1-\alpha'_n-\gamma'_n)x_n + \gamma'_n S^n z_n; \\ z_n = P_C(I-\lambda A)u_n; \\ u_n = P_C(I-\mu B)x_n, \end{cases} \tag{3.42}$$

where $0 < \lambda < \frac{\xi}{K^2}, 0 < \mu < \frac{\eta}{K^2}$. Suppose $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\alpha'_n\}, \{\beta'_n\}, \{\gamma'_n\}$ are sequences in $[0,1]$ satisfying the following conditions:

(a) $0 < \alpha'_n \le \alpha_n \le \gamma'_n \le \gamma_n < 1$;
(b) $0 < \liminf \beta_n \le \limsup \beta_n < 1$;
(c) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty, \sum_{n=1}^{\infty} \gamma_n^2 < \infty, \lim_{n\to\infty} \frac{k_n - 1}{\alpha_n} = 0$;
(d) *T, S satisfy the asymptotically regularity:*
$\lim_{n\to\infty} \|T^{n+1}x_n - T^n x_n\| = 0 = \lim_{n\to\infty} \|S^{n+1}x_n - S^n x_n\|$.

Then, the sequence $\{x_n\}$ converges strongly to the unique solution of the variational inequality: $q \in \mathcal{F}$ such that

$$\langle (I-f)q, j(q-x^\star)\rangle \leq 0, \forall x^\star \in \mathcal{F}.$$

Proof. Since H is a Hilbert space, then 2-uniformly smooth constant $K = \dfrac{\sqrt{2}}{2}$. The result follows from Theorem 3.1. □

Remark 3.1. Since every asymptotically nonexpansive mapping is a superclass of the class of nonexpansive mapping, the above results remain valid when S, T are nonexpansive mappings.

4. Application

Here, we give an application of our main result to a variational inequality problem for two strictly pseudocontractive mappings in Hilbert space.

Theorem 4.1. *Let C be a nonempty closed convex subset of of a Hilbert space H. Let P_C be a metric projection from H onto C. Let $S^\star, T^\star : C \longrightarrow C$ be k_1-strictly pseudocontractive and k_2-strictly pseudocontractive self mappings on C. Let $f : C \longrightarrow C$ be a ρ-strict contraction of C into itself with coefficient $\rho \in (0,1)$. Let $S, T : C \longrightarrow C$ be asymptotically nonexpansive self mappings on C such that $\mathcal{F} = F(T)\cap F(S)\cap F(G) \neq \emptyset$, where G is as defined by Lemma 2.9. For arbitrarily chosen $x_1 \in C$, let the sequence $\{x_n\}_{n\geq 1}$ be defined iteratively as follows:*

$$\begin{cases} x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n-\gamma_n)x_n + \gamma_n T^n y_n; \\ y_n = \alpha'_n f(x_n) + (1-\alpha'_n-\gamma'_n)x_n + \gamma'_n S^n z_n; \\ z_n = (1-\lambda)u_n + \lambda T^\star u_n; \\ u_n = (1-\mu)x_n + \mu S^\star x_n, \end{cases} \tag{4.1}$$

where $0 < \lambda < (1-k_1), 0 < \mu < (1-k_2)$. Suppose $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\alpha'_n\}, \{\beta'_n\}, \{\gamma'_n\}$ are sequences in $[0,1]$ satisfying the following conditions:

(a) $0 < \alpha'_n \leq \alpha_n \leq \gamma'_n \leq \gamma_n < 1$;
(b) $0 < \liminf \beta_n \leq \limsup \beta_n < 1$;
(c) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^\infty \alpha_n = \infty, \sum_{n=1}^\infty \gamma_n^2 < \infty, \lim_{n\to\infty} \dfrac{k_n - 1}{\alpha_n} = 0$;
(d) *T, S satisfy the asymptotically regularity:* $\lim_{n\to\infty} \|T^{n+1}x_n - T^n x_n\| = 0 = \lim_{n\to\infty} \|S^{n+1}x_n - S^n x_n\|$.

Then, the sequence $\{x_n\}$ converges strongly to the unique solution of the variational inequality: $q \in \mathcal{F}$ such that

$$\langle (I-f)q, j(q-x^\star)\rangle \leq 0, \forall x^\star \in \mathcal{F}.$$

Proof. Let $A = I - T^\star : C \longrightarrow H$ and $B = I - S^\star : C \longrightarrow H$. Then, by Lemma 2.13, $A : C \longrightarrow H$ is ξ-inverse-strongly accretive with $\xi = \dfrac{1-k_1}{2}$ and $B : C \longrightarrow H$ is η-inverse-strongly accretive with $\eta = \dfrac{1-k_2}{2}$.

Also, observe that

$$\begin{aligned} z_n &= P_C(u_n - \lambda A u_n) \\ &= P_C(u_n - \lambda(I - T^\star)u_n) \\ &= P_C((1-\lambda)u_n + \lambda T^\star u_n) \\ &= (1-\lambda)u_n + \lambda T^\star u_n \end{aligned}$$

and

$$\begin{aligned} u_n &= P_C(x_n - \mu B x_n) \\ &= P_C(x_n - \mu(I - S^\star)x_n) \\ &= P_C((1-\mu)x_n + \mu S^\star X_n) \\ &= (1-\mu)x_n + \mu S^\star x_n \end{aligned}$$

Using the above information and Corollary 3.1, the proof of Theorem 4.1 is completed. □

Acknowledgements. The authors thank the anonymous reviewers for their careful reading of this paper and approved the final manuscript.

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