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DIFFERENTIAL SANDWICH THEOREMS FOR BAZILEVIČ FUNCTION DEFINED BY CONVOLUTION STRUCTURE

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Abstract. In this present paper, we obtain some applications of first-order differential subordination and superordination results involving Hadamard product for multivalent analytic functions with generalized hypergeometric function in the open unit disk. These results are applied to obtain sandwich results.

1. Introduction and Preliminaries

Denote by $\mathcal{H}$ the collection of analytic functions in the unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ and assume that $\mathcal{H}[a, n]$ be the subclass of $\mathcal{H}$ consisting of functions of the form:

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + ... \quad (a \in \mathbb{C},\ n \in \mathbb{N} = \{1, 2, ...\}).$$

Also, let $\mathcal{A}$ be the subclass of $\mathcal{H}$ consisting of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

A function $f \in \mathcal{A}$ is called Bazilevič function, if it satisfies the condition

$$\Re\left\{\frac{z^{1-\lambda} f'(z)}{f^{1-\lambda}(z)}\right\} > 0,\ (0 \leqq \lambda \leqq 1, z \in \mathbb{U}).$$

This class of functions was denoted by B_λ $(0 \leqq \lambda \leqq 1)$ and studied by Singh [6].

For the functions $f \in \mathcal{A}$ given by (1.1) and $g \in \mathcal{A}$ defined by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n,$$

we define the Hadamard product (or convolution) $f * g$ of the functions f and g (as usual) by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n = (g * f)(z).$$

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Now we recall the principle of subordination between analytic functions, let the functions f and g be analytic in $\mathbb{U}$. We say that the function f is subordinate to g, if there exists a Schwarz function ω, which is analytic in $\mathbb{U}$ with

$$\omega(0) = 0 \qquad \text{and} \qquad |\omega(z)| < 1 \quad (z \in \mathbb{U}),$$

such that

$$f(z) = g(\omega(z)).$$

This subordination is indicated by

$$f \prec g \qquad \text{or} \qquad f(z) \prec g(z) \quad (z \in \mathbb{U}).$$

It is well known that (see [3]), if the function g is univalent in $\mathbb{U}$, then

$$f \prec g \quad (z \in \mathbb{U}) \iff f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subseteq g(\mathbb{U}).$$

Let $k, h \in \mathcal{H}$ and $\psi(r, s; z) : \mathbb{C}^2 \times \mathbb{U} \to \mathbb{C}$. If k and $\psi(k(z), zk'(z), z^2 k''(z); z)$ are univalent functions in $\mathbb{U}$ and if k satisfies the first-order differential superordination

$$h(z) \prec \psi(k(z), zk'(z); z), \tag{1.2}$$

then k is called a solution of the differential superordination (1.2). (If f is subordinate to g, then g is superordinate to f). An analytic function q is called a subordinate of (1.2), if $q \prec k$ for all the functions k satisfying (1.2). An univalent subordinat $\check{q}$ that satisfies $q \prec \check{q}$ for all the subordinants q of (1.2) is called the best subordinant.

Very recently many authors, Rahrovi [4], Attiya and Yassen [1], Seoudy [5] and Wanas and Majeed [7] have obtained sandwich results for certain classes of analytic functions.

The main object of the present work is to find sufficient condition for certain normalized analytic functions f in $\mathbb{U}$ such that $(f * \Psi)(z) \neq 0$ and f to satisfy

$$q_1(z) \prec \left(\frac{z^{1-\lambda} (f * \Phi)'(z)}{((f * \Psi)(z))^{1-\lambda}} \right)^{\gamma} \prec q_2(z)$$

and

$$q_1(z) \prec \left(1 + \frac{z^{2-\lambda} (f * \Phi)''(z)}{\left(z (f * \Psi)'(z) \right)^{1-\lambda}} \right)^{\gamma} \prec q_2(z),$$

where q_1 and q_2 are given univalent functions in $\mathbb{U}$ with $q_1(0) = q_2(0) = 1$ and

$$\Phi(z) = z + \sum_{n=2}^{\infty} r_n z^n, \;\; \Psi(z) = z + \sum_{n=2}^{\infty} e_n z^n$$

are analytic functions in $\mathbb{U}$ with $r_n \geqq 0, e_n \geqq 0$. We obtain number of known results as special cases.

To prove our main results, we will require the following definition and lemmas.

Definition 1.1. [3] Denote by Q the set of all functions f that are analytic and injective on $\bar{\mathbb{U}} \backslash E(f)$, where

$$E(f) = \left\{ \zeta \in \partial\mathbb{U} : \lim_{z \to \zeta} f(z) = \infty \right\}$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial\mathbb{U} \backslash E(f)$.

Lemma 1.1. [3] *Let q be univalent in the unite disk $\mathbb{U}$ and let θ and ϕ be analytic in a domain D containing $q(\mathbb{U})$ with $\phi(w) \neq 0$ when $w \in q(\mathbb{U})$. set $Q(z) = zq'(z)\phi(q(z))$ and $h(z) = \theta(q(z)) + Q(z)$. Suppose that*
(1)$Q(z)$ is starlike univalent in $\mathbb{U}$,
(2)$\Re\left\{\frac{zh'(z)}{Q(z)}\right\} > 0$ for $z \in \mathbb{U}$.
If k is analytic in $\mathbb{U}$, with $k(0) = q(0)$, $k(\mathbb{U}) \subset D$ and

$$\theta(k(z)) + zk'(z)\phi(k(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)), \tag{1.3}$$

then $k \prec q$ and q is the best dominant of (1.3).

Lemma 1.2. [2] *Let q be convex univalent in the unit disk $\mathbb{U}$ and let θ and ϕ be analytic in a domain D containing $q(\mathbb{U})$. Suppose that*
(1)$\Re\left\{\frac{\theta'(q(z))}{\phi(q(z))}\right\} > 0$ for $z \in \mathbb{U}$,
(2)$Q(z) = zq'(z)\phi(q(z))$ is starlike univalent in $\mathbb{U}$.
If $k \in \mathcal{H}[q(0), 1] \cap Q$, with $k(\mathbb{U}) \subset D$, $\theta(k(z)) + zk'(z)\phi(k(z))$ is univalent in $\mathbb{U}$ and

$$\theta(q(z)) + zq'(z)\phi(q(z)) \prec \theta(k(z)) + zk'(z)\phi(k(z)), \tag{1.4}$$

then $q \prec k$ and q is the best subordinant of (1.4).

2. Subordination Results

Theorem 2.1. *Let $\Phi, \Psi \in \mathcal{A}$, $\rho, \delta, \eta, \mu, \gamma \in \mathbb{C}$ such that $\gamma \neq 0$ and q be convex univalent in $\mathbb{U}$ with $q(0) = 1$ and assume that*

$$\Re\left\{1 + \frac{\delta q^2(z) - \eta}{\mu q(z)} + \frac{zq''(z)}{q'(z)}\right\} > 0. \tag{2.1}$$

If $f \in \mathcal{A}$ satisfies the differential subordination

$$\Omega_1(f, \Phi, \Psi, \rho, \delta, \eta, \mu, \gamma, \lambda; z) \prec \rho + \delta q(z) + \frac{\eta}{q(z)} + \mu\frac{zq'(z)}{q(z)}, \tag{2.2}$$

where

$$\begin{aligned}\Omega_1(f, \Phi, \Psi, \rho, \delta, \eta, \mu, \gamma, \lambda; z) &= \rho + \delta\left(\frac{z^{1-\lambda}(f * \Phi)'(z)}{((f * \Psi)(z))^{1-\lambda}}\right)^{\gamma} + \eta\left(\frac{((f * \Psi)(z))^{1-\lambda}}{z^{1-\lambda}(f * \Phi)'(z)}\right)^{\gamma} \\ &+ \gamma\mu\left[\frac{z(f * \Phi)''(z)}{(f * \Phi)'(z)} + (1 - \lambda)\left(1 - \frac{z(f * \Psi)'(z)}{(f * \Psi)(z)}\right)\right],\end{aligned} \tag{2.3}$$

then

$$\left(\frac{z^{1-\lambda}(f * \Phi)'(z)}{((f * \Psi)(z))^{1-\lambda}}\right)^{\gamma} \prec q(z)$$

and q is the best dominant of (2.2).

Proof. Let us define

$$k(z) = \left(\frac{z^{1-\lambda}(f * \Phi)'(z)}{((f * \Psi)(z))^{1-\lambda}}\right)^{\gamma}, \quad (z \in \mathbb{U}). \tag{2.4}$$

Then the function k is analytic in $\mathbb{U}$ and $k(0) = 1$.
By setting
$$\theta(w) = \rho + \delta w + \frac{\eta}{w} \quad \textit{and} \quad \phi(w) = \frac{\mu}{w},$$
it can be easily observed that $\theta(w)$ and $\phi(w)$ are analytic in $\mathbb{C}\backslash\{0\}$ and that $\phi(w) \neq 0$, $w \in \mathbb{C}\backslash\{0\}$. Also, we get
$$Q(z) = zq'(z)\phi(q(z)) = \mu\frac{zq'(z)}{q(z)}$$
and
$$h(z) = \theta(q(z)) + Q(z) = \rho + \delta q(z) + \frac{\eta}{q(z)} + \mu\frac{zq'(z)}{q(z)}.$$
In light of the hypothesis of Theorem 2.1, we see that $Q(z)$ is starlike univalent in $\mathbb{U}$ and
$$\Re\left\{\frac{zh'(z)}{Q(z)}\right\} = \Re\left\{1 + \frac{\delta q^2(z) - \eta}{\mu q(z)} + \frac{zq''(z)}{q'(z)}\right\} > 0.$$
A simple computation using (2.4) gives
$$\frac{zk'(z)}{k(z)} = \gamma\left[\frac{z\left(f * \Phi\right)''(z)}{\left(f * \Phi\right)'(z)} + (1-\lambda)\left(1 - \frac{z\left(f * \Psi\right)'(z)}{\left(f * \Psi\right)(z)}\right)\right].$$
Also, we find that
$$\rho + \delta k(z) + \frac{\eta}{k(z)} + \mu\frac{zk'(z)}{k(z)} = \Omega_1(f, \Phi, \Psi, \rho, \delta, \eta, \mu, \gamma, \lambda; z), \tag{2.5}$$
where $\Omega_1(f, \Phi, \Psi, \rho, \delta, \eta, \mu, \gamma, \lambda; z)$ is given by (2.3).
By using (2.5) in (2.2), we deduce that
$$\rho + \delta k(z) + \frac{\eta}{k(z)} + \mu\frac{zk'(z)}{k(z)} \prec \rho + \delta q(z) + \frac{\eta}{q(z)} + \mu\frac{zq'(z)}{q(z)}.$$
Hence by an application of Lemma 1.1, we have $p(z) \prec q(z)$. By using (2.4), we obtain the result which we needed. □

By fixing $\Phi(z) = \Psi(z) = \frac{z}{1-z}$ in Theorem 2.1, we obtain the following Corollary:

Corollary 2.1. *Let $\rho, \delta, \eta, \mu, \gamma \in \mathbb{C}$ such that $\gamma \neq 0$ and q be convex univalent in $\mathbb{U}$ with $q(0) = 1$ and assume that (2.1) holds true. If $f \in \mathcal{A}$ satisfies the differential subordination*
$$\Omega_2(f, \rho, \delta, \eta, \mu, \gamma, \lambda; z) \prec \rho + \delta q(z) + \frac{\eta}{q(z)} + \mu\frac{zq'(z)}{q(z)}, \tag{2.6}$$
where
$$\Omega_2(f, \rho, \delta, \eta, \mu, \gamma, \lambda; z) = \rho + \delta\left(\frac{z^{1-\lambda}f'(z)}{(f(z))^{1-\lambda}}\right)^{\gamma} + \eta\left(\frac{(f(z))^{1-\lambda}}{z^{1-\lambda}f'(z)}\right)^{\gamma}$$
$$+ \gamma\mu\left[\frac{zf''(z)}{f'(z)} + (1-\lambda)\left(1 - \frac{zf'(z)}{f(z)}\right)\right], \tag{2.7}$$
then
$$\left(\frac{z^{1-\lambda}f'(z)}{(f(z))^{1-\lambda}}\right)^{\gamma} \prec q(z)$$
and q is the best dominant of (2.6).

By taking $\lambda = 0$ in Theorem 2.1, we obtain the following corollary:

Corollary 2.2. *Let $\Phi, \Psi \in \mathcal{A}$, $\rho, \delta, \eta, \mu, \gamma \in \mathbb{C}$ such that $\gamma \neq 0$ and q be convex univalent in $\mathbb{U}$ with $q(0) = 1$ and assume that (2.1) holds true. If $f \in \mathcal{A}$ satisfies the differential subordination*

$$\Omega_3(f, \Phi, \Psi, \rho, \delta, \eta, \mu, \gamma; z) \prec \rho + \delta q(z) + \frac{\eta}{q(z)} + \mu \frac{zq'(z)}{q(z)}, \tag{2.8}$$

where

$$\begin{aligned}\Omega_3(f, \Phi, \Psi, \rho, \delta, \eta, \mu, \gamma; z) &= \rho + \delta \left(\frac{z\left(f * \Phi\right)'(z)}{\left(f * \Psi\right)(z)} \right)^{\gamma} + \eta \left(\frac{\left(f * \Psi\right)(z)}{z\left(f * \Phi\right)'(z)} \right)^{\gamma} \\ &\quad + \gamma\mu \left[1 + \frac{z\left(f * \Phi\right)''(z)}{\left(f * \Phi\right)'(z)} - \frac{z\left(f * \Psi\right)'(z)}{\left(f * \Psi\right)(z)} \right],\end{aligned} \tag{2.9}$$

then

$$\left(\frac{z\left(f * \Phi\right)'(z)}{\left(f * \Psi\right)(z)} \right)^{\gamma} \prec q(z)$$

and q is the best dominant of (2.8).

Theorem 2.2. *Let $\Phi, \Psi \in \mathcal{A}$, $\rho, \delta, \eta, \mu, \gamma \in \mathbb{C}$ such that $\gamma \neq 0$ and q be convex univalent in $\mathbb{U}$ with $q(0) = 1$ and assume that (2.1) holds true. If $f \in \mathcal{A}$ satisfies the differential subordination*

$$\Omega_4(f, \Phi, \Psi, \rho, \delta, \eta, \mu, \gamma, \lambda; z) \prec \rho + \delta q(z) + \frac{\eta}{q(z)} + \mu \frac{zq'(z)}{q(z)}, \tag{2.10}$$

where

$$\begin{aligned}&\Omega_4(f, \Phi, \Psi, \rho, \delta, \eta, \mu, \gamma, \lambda; z) \\ &= \rho + \delta \left(1 + \frac{z^{2-\lambda}\left(f * \Phi\right)''(z)}{\left(z\left(f * \Psi\right)'(z)\right)^{1-\lambda}} \right)^{\gamma} + \eta \left(\frac{\left(z\left(f * \Psi\right)'(z)\right)^{1-\lambda}}{\left(z\left(f * \Psi\right)'(z)\right)^{1-\lambda} + z^{2-\lambda}\left(f * \Phi\right)''(z)} \right)^{\gamma} \\ &+ \gamma\mu \left[\frac{z\left(f * \Phi\right)'''(z)}{\left(f * \Phi\right)''(z)} + (1-\lambda)\frac{z\left(f * \Psi\right)''(z)}{\left(f * \Psi\right)'(z)} + 3 - 2\lambda \right],\end{aligned} \tag{2.11}$$

then

$$\left(1 + \frac{z^{2-\lambda}\left(f * \Phi\right)''(z)}{\left(z\left(f * \Psi\right)'(z)\right)^{1-\lambda}} \right)^{\gamma} \prec q(z)$$

and q is the best dominant of (2.10).

Proof. Let us define

$$k(z) = \left(1 + \frac{z^{2-\lambda}\left(f * \Phi\right)''(z)}{\left(z\left(f * \Psi\right)'(z)\right)^{1-\lambda}} \right)^{\gamma}, \quad (z \in \mathbb{U}). \tag{2.12}$$

Then the function k is analytic in $\mathbb{U}$ and $k(0) = 1$.
After some calculations from (2.12), we conclude that

$$\rho + \delta k(z) + \frac{\eta}{k(z)} + \mu \frac{zk'(z)}{k(z)} = \Omega_4(f, \Phi, \Psi, \rho, \delta, \eta, \mu, \gamma, \lambda; z), \tag{2.13}$$

where $\Omega_4(f, \Phi, \Psi, \rho, \delta, \eta, \mu, \gamma, \lambda; z)$ is given by (2.11).
In view of (2.13), the subordination (2.10), can be written as

$$\rho + \delta k(z) + \frac{\eta}{k(z)} + \mu \frac{zk'(z)}{k(z)} \prec \rho + \delta q(z) + \frac{\eta}{q(z)} + \mu \frac{zq'(z)}{q(z)}.$$

By setting $\theta(w) = \rho + \delta w + \frac{\eta}{w}$ and $\phi(w) = \frac{\mu}{w}$, it is easily observed that $\theta(w)$ and $\phi(w)$ are analytic in $\mathbb{C}\backslash\{0\}$ and that $\phi(w) \neq 0$, $w \in \mathbb{C}\backslash\{0\}$. Hence the result now follows by an application of Lemma 1.1. □

By fixing $\Phi(z) = \Psi(z) = \frac{z}{1-z}$ in Theorem 2.2, we obtain the following corollary:

Corollary 2.3. *Let $\rho, \delta, \eta, \mu, \gamma \in \mathbb{C}$ such that $\gamma \neq 0$ and q be convex univalent in $\mathbb{U}$ with $q(0) = 1$ and assume that (2.1) holds true. If $f \in \mathcal{A}$ satisfies the differential subordination*

$$\Omega_5(f, \rho, \delta, \eta, \mu, \gamma, \lambda; z) \prec \rho + \delta q(z) + \frac{\eta}{q(z)} + \mu \frac{zq'(z)}{q(z)}, \tag{2.14}$$

where

$$\begin{aligned} \Omega_5(f, \rho, \delta, \eta, \mu, \gamma, \lambda; z) = \rho + \delta \left(1 + \frac{z^{2-\lambda} f''(z)}{(zf'(z))^{1-\lambda}}\right)^{\gamma} + \eta \left(\frac{(zf'(z))^{1-\lambda}}{(zf'(z))^{1-\lambda} + z^{2-\lambda} f''(z)}\right)^{\gamma} \\ + \gamma\mu \left[\frac{zf'''(z)}{f''(z)} + (1-\lambda)\frac{zf''(z)}{f'(z)} + 3 - 2\lambda\right], \end{aligned} \tag{2.15}$$

then

$$\left(1 + \frac{z^{2-\lambda} f''(z)}{(zf'(z))^{1-\lambda}}\right)^{\gamma} \prec q(z)$$

and q is the best dominant of (2.14).

By taking $\lambda = 0$ in Theorem 2.2, we obtain the following corollary:

Corollary 2.4. *Let $\Phi, \Psi \in \mathcal{A}$, $\rho, \delta, \eta, \mu, \gamma \in \mathbb{C}$ such that $\gamma \neq 0$ and q be convex univalent in $\mathbb{U}$ with $q(0) = 1$ and assume that (2.1) holds true. If $f \in \mathcal{A}$ satisfies the differential subordination*

$$\Omega_6(f, \Phi, \Psi, \rho, \delta, \eta, \mu, \gamma; z) \prec \rho + \delta q(z) + \frac{\eta}{q(z)} + \mu \frac{zq'(z)}{q(z)}, \tag{2.16}$$

where

$$\begin{aligned} \Omega_6(f, \Phi, \Psi, \rho, \delta, \eta, \mu, \gamma; z) = \rho + \delta \left(1 + \frac{z\,(f*\Phi)''(z)}{(f*\Psi)'(z)}\right)^{\gamma} + \eta \left(\frac{(f*\Psi)'(z)}{(f*\Psi)'(z) + z\,(f*\Phi)''(z)}\right)^{\gamma} \\ + \gamma\mu \left[\frac{z\,(f*\Phi)'''(z)}{(f*\Phi)''(z)} + \frac{z\,(f*\Psi)''(z)}{(f*\Psi)'(z)} + 3\right], \end{aligned} \tag{2.17}$$

then

$$\left(1+\frac{z\left(f*\Phi\right)''(z)}{\left(f*\Psi\right)'(z)}\right)^{\gamma}\prec q(z)$$

and q is the best dominant of (2.16).

3. Superordination Results

Theorem 3.1. *Let $\Phi,\Psi\in\mathcal{A}$, $\rho,\delta,\eta,\mu,\gamma\in\mathbb{C}$ such that $\gamma\neq 0$ and q be convex univalent in $\mathbb{U}$ with $q(0)=1$ and assume that*

$$\Re\left\{\frac{\left(\delta q^{2}(z)-\eta\right)q'(z)}{\mu q(z)}\right\}>0. \tag{3.1}$$

Suppose that $f\in\mathcal{A}$, $\left(\frac{z^{1-\lambda}(f*\Phi)'(z)}{((f*\Psi)(z))^{1-\lambda}}\right)^{\gamma}\in\mathcal{H}\left[q(0),1\right]\cap Q$ and $\Omega_1(f,\Phi,\Psi,\rho,\delta,\eta,\mu,\gamma,\lambda;z)$ as defined by (2.3) be univalent in $\mathbb{U}$. If

$$\rho+\delta q(z)+\frac{\eta}{q(z)}+\mu\frac{zq'(z)}{q(z)}\prec\Omega_1(f,\Phi,\Psi,\rho,\delta,\eta,\mu,\gamma,\lambda;z), \tag{3.2}$$

then

$$q(z)\prec\left(\frac{z^{1-\lambda}\left(f*\Phi\right)'(z)}{\left(\left(f*\Psi\right)(z)\right)^{1-\lambda}}\right)^{\gamma}$$

and q is the best subordinant of (3.2).

Proof. Let the function k be defined by (2.4). By a straightforward computation, the superordination (3.2) becomes

$$\rho+\delta q(z)+\frac{\eta}{q(z)}+\mu\frac{zq'(z)}{q(z)}\prec\rho+\delta k(z)+\frac{\eta}{k(z)}+\mu\frac{zk'(z)}{k(z)}.$$

By setting $\theta(w)=\rho+\delta w+\frac{\eta}{w}$ and $\phi(w)=\frac{\mu}{w}$, it is easily observed that $\theta(w)$ and $\phi(w)$ are analytic in $\mathbb{C}\backslash\{0\}$ and that $\phi(w)\neq 0$, $w\in\mathbb{C}\backslash\{0\}$. Also, we have

$$\Re\left\{\frac{\theta'(q(z))}{\phi(q(z))}\right\}=\Re\left\{\frac{\left(\delta q^{2}(z)-\eta\right)q'(z)}{\mu q(z)}\right\}>0.$$

Now Theorem 3.1 follows by applying Lemma 1.2. □

By fixing $\Phi(z)=\Psi(z)=\frac{z}{1-z}$ in Theorem 3.1, we obtain the following corollary:

Corollary 3.1. *Let $\rho,\delta,\eta,\mu,\gamma\in\mathbb{C}$ such that $\gamma\neq 0$ and q be convex univalent in $\mathbb{U}$ with $q(0)=1$ and assume that (3.1) holds true. Suppose that $f\in\mathcal{A}$,*

$$\left(\frac{z^{1-\lambda}f'(z)}{\left(f(z)\right)^{1-\lambda}}\right)^{\gamma}\in\mathcal{H}\left[q(0),1\right]\cap Q$$

and $\Omega_2(f,\rho,\delta,\eta,\mu,\gamma,\lambda;z)$ as defined by (2.7) be univalent in $\mathbb{U}$. If

$$\rho+\delta q(z)+\frac{\eta}{q(z)}+\mu\frac{zq'(z)}{q(z)}\prec\Omega_2(f,\rho,\delta,\eta,\mu,\gamma,\lambda;z), \tag{3.3}$$

then

$$q(z) \prec \left(\frac{z^{1-\lambda}f'(z)}{(f(z))^{1-\lambda}}\right)^{\gamma}$$

and q is the best subordinant of (3.3).

By taking $\lambda = 0$ in Theorem 3.1, we obtain the following corollary:

Corollary 3.2. *Let $\Phi, \Psi \in \mathcal{A}$, $\rho, \delta, \eta, \mu, \gamma \in \mathbb{C}$ such that $\gamma \neq 0$ and q be convex univalent in $\mathbb{U}$ with $q(0) = 1$ and assume that (3.1) holds true. Suppose that $f \in \mathcal{A}$,*

$$\left(\frac{z\,(f*\Phi)'\,(z)}{(f*\Psi)\,(z)}\right)^{\gamma} \in \mathcal{H}\,[q(0), 1] \cap Q$$

and $\Omega_3(f, \Phi, \Psi, \rho, \delta, \eta, \mu, \gamma; z)$ as defined by (2.9) be univalent in $\mathbb{U}$. If

$$\rho + \delta q(z) + \frac{\eta}{q(z)} + \mu\frac{zq'(z)}{q(z)} \prec \Omega_3(f, \Phi, \Psi, \rho, \delta, \eta, \mu, \gamma; z), \tag{3.4}$$

then

$$q(z) \prec \left(\frac{z\,(f*\Phi)'\,(z)}{(f*\Psi)\,(z)}\right)^{\gamma}$$

and q is the best subordinant of (3.4).

Theorem 3.2. *Let $\Phi, \Psi \in \mathcal{A}$, $\rho, \delta, \eta, \mu, \gamma \in \mathbb{C}$ such that $\gamma \neq 0$ and q be convex univalent in $\mathbb{U}$ with $q(0) = 1$ and assume that (3.1) holds true. Suppose that $f \in \mathcal{A}$,*

$$\left(1 + \frac{z^{2-\lambda}\,(f*\Phi)''\,(z)}{\left(z\,(f*\Psi)'\,(z)\right)^{1-\lambda}}\right)^{\gamma} \in \mathcal{H}\,[q(0), 1] \cap Q$$

and $\Omega_4(f, \Phi, \Psi, \rho, \delta, \eta, \mu, \gamma, \lambda; z)$ as defined by (2.11) be univalent in $\mathbb{U}$. If

$$\rho + \delta q(z) + \frac{\eta}{q(z)} + \mu\frac{zq'(z)}{q(z)} \prec \Omega_4(f, \Phi, \Psi, \rho, \delta, \eta, \mu, \gamma, \lambda; z), \tag{3.5}$$

then

$$q(z) \prec \left(1 + \frac{z^{2-\lambda}\,(f*\Phi)''\,(z)}{\left(z\,(f*\Psi)'\,(z)\right)^{1-\lambda}}\right)^{\gamma}$$

and q is the best subordinant of (3.5).

For the choice of $k(z) = \left(1 + \frac{z^{2-\lambda}(f*\Phi)''(z)}{\left(z(f*\Psi)'(z)\right)^{1-\lambda}}\right)^{\gamma}$, the proof of Theorem 3.2 is line similar to the proof of Theorem 3.1 and hence we omit it.

By fixing $\Phi(z) = \Psi(z) = \frac{z}{1-z}$ in Theorem 3.2, we obtain the following corollary:

Corollary 3.3. *Let $\rho, \delta, \eta, \mu, \gamma \in \mathbb{C}$ such that $\gamma \neq 0$ and q be convex univalent in $\mathbb{U}$ with $q(0) = 1$ and assume that (3.1) holds true. Suppose that $f \in \mathcal{A}$,*

$$\left(1 + \frac{z^{2-\lambda}f''(z)}{(zf'(z))^{1-\lambda}}\right)^{\gamma} \in \mathcal{H}\,[q(0), 1] \cap Q$$

and $\Omega_5(f,\rho,\delta,\eta,\mu,\gamma,\lambda;z)$ as defined by (2.15) be univalent in $\mathbb{U}$. If

$$\rho+\delta q(z)+\frac{\eta}{q(z)}+\mu\frac{zq'(z)}{q(z)}\prec\Omega_5(f,\rho,\delta,\eta,\mu,\gamma,\lambda;z), \tag{3.6}$$

then

$$q(z)\prec\left(1+\frac{z^{2-\lambda}f''(z)}{(zf'(z))^{1-\lambda}}\right)^{\gamma}$$

and q is the best subordinant of (3.6).

By taking $\lambda=0$ in Theorem 3.2, we obtain the following corollary:

Corollary 3.4. *Let $\Phi,\Psi\in\mathcal{A}$, $\rho,\delta,\eta,\mu,\gamma\in\mathbb{C}$ such that $\gamma\neq0$ and q be convex univalent in $\mathbb{U}$ with $q(0)=1$ and assume that (3.1) holds true. Suppose that $f\in\mathcal{A}$,*

$$\left(1+\frac{z\,(f*\Phi)''\,(z)}{(f*\Psi)'\,(z)}\right)^{\gamma}\in\mathcal{H}\left[q(0),1\right]\cap Q$$

and $\Omega_6(f,\Phi,\Psi,\rho,\delta,\eta,\mu,\gamma;z)$ as defined by (2.17) be univalent in $\mathbb{U}$. If

$$\rho+\delta q(z)+\frac{\eta}{q(z)}+\mu\frac{zq'(z)}{q(z)}\prec\Omega_6(f,\Phi,\Psi,\rho,\delta,\eta,\mu,\gamma;z), \tag{3.7}$$

then

$$q(z)\prec\left(1+\frac{z\,(f*\Phi)''\,(z)}{(f*\Psi)'\,(z)}\right)^{\gamma}$$

and q is the best subordinant of (3.7).

4. Sandwich Results

Concluding the results of differential subordination and superordination, we arrive at the following "sandwich results".

Theorem 4.1. *Let q_1 and q_2 be convex univalent in $\mathbb{U}$ with $q_1(0)=q_2(0)=1$, $\rho,\delta,\eta,\mu,\gamma\in\mathbb{C}$ such that $\gamma\neq0$ and let q_2 satisfies (2.1) and q_1 satisfies (3.1). For $f,\Phi,\Psi\in\mathcal{A}$, let*

$$\left(\frac{z^{1-\lambda}\,(f*\Phi)'\,(z)}{((f*\Psi)\,(z))^{1-\lambda}}\right)^{\gamma}\in\mathcal{H}\left[1,1\right]\cap Q$$

and $\Omega_1(f,\Phi,\Psi,\rho,\delta,\eta,\mu,\gamma,\lambda;z)$ as defined by (2.3) be univalent in $\mathbb{U}$. If

$$\rho+\delta q_1(z)+\frac{\eta}{q_1(z)}+\mu\frac{zq_1'(z)}{q_1(z)}\prec\Omega_1(f,\Phi,\Psi,\rho,\delta,\eta,\mu,\gamma,\lambda;z)$$
$$\prec\rho+\delta q_2(z)+\frac{\eta}{q_2(z)}+\mu\frac{zq_2'(z)}{q_2(z)},$$

then

$$q_1(z)\prec\left(\frac{z^{1-\lambda}\,(f*\Phi)'\,(z)}{((f*\Psi)\,(z))^{1-\lambda}}\right)^{\gamma}\prec q_2(z)$$

and q_1, q_2 are respectively the best subordinant and the best dominant.

Theorem 4.2. *Let q_1 and q_2 be convex univalent in $\mathbb{U}$ with $q_1(0) = q_2(0) = 1$, $\rho, \delta, \eta, \mu, \gamma \in \mathbb{C}$ such that $\gamma \neq 0$ and let q_2 satisfies (2.1) and q_1 satisfies (3.1). For $f, \Phi, \Psi \in \mathcal{A}$, let*

$$\left(1 + \frac{z^{2-\lambda} (f * \Phi)'' (z)}{\left(z (f * \Psi)' (z)\right)^{1-\lambda}}\right)^{\gamma} \in \mathcal{H}[1, 1] \cap Q$$

and $\Omega_4(f, \Phi, \Psi, \rho, \delta, \eta, \mu, \gamma, \lambda; z)$ as defined by (2.11) be univalent in $\mathbb{U}$. If

$$\rho + \delta q_1(z) + \frac{\eta}{q_1(z)} + \mu \frac{z q_1'(z)}{q_1(z)} \prec \Omega_4(f, \Phi, \Psi, \rho, \delta, \eta, \mu, \gamma, \lambda; z)$$
$$\prec \rho + \delta q_2(z) + \frac{\eta}{q_2(z)} + \mu \frac{z q_2'(z)}{q_2(z)},$$

then

$$q_1(z) \prec \left(1 + \frac{z^{2-\lambda} (f * \Phi)'' (z)}{\left(z (f * \Psi)' (z)\right)^{1-\lambda}}\right)^{\gamma} \prec q_2(z)$$

and q_1, q_2 are respectively the best subordinant and the best dominant.

By making use of Corollaries 2.1 and 3.1, we obtain the following corollary:

Corollary 4.1. *Let q_1 and q_2 be convex univalent in $\mathbb{U}$ with $q_1(0) = q_2(0) = 1$, $\rho, \delta, \eta, \mu, \gamma \in \mathbb{C}$ such that $\gamma \neq 0$ and let q_2 satisfies (2.1) and q_1 satisfies (3.1). For $f \in \mathcal{A}$, let*

$$\left(\frac{z^{1-\lambda} f'(z)}{(f(z))^{1-\lambda}}\right)^{\gamma} \in \mathcal{H}[1, 1] \cap Q$$

and $\Omega_2(f, \rho, \delta, \eta, \mu, \gamma, \lambda; z)$ as defined by (2.7) be univalent in $\mathbb{U}$. If

$$\rho + \delta q_1(z) + \frac{\eta}{q_1(z)} + \mu \frac{z q_1'(z)}{q_1(z)} \prec \Omega_2(f, \rho, \delta, \eta, \mu, \gamma, \lambda; z) \prec \rho + \delta q_2(z) + \frac{\eta}{q_2(z)} + \mu \frac{z q_2'(z)}{q_2(z)},$$

then

$$q_1(z) \prec \left(\frac{z^{1-\lambda} f'(z)}{(f(z))^{1-\lambda}}\right)^{\gamma} \prec q_2(z)$$

and q_1, q_2 are respectively the best subordinant and the best dominant.

By making use of Corollaries 2.2 and 3.2, we obtain the following corollary:

Corollary 4.2. *Let q_1 and q_2 be convex univalent in $\mathbb{U}$ with $q_1(0) = q_2(0) = 1$, $\rho, \delta, \eta, \mu, \gamma \in \mathbb{C}$ such that $\gamma \neq 0$ and let q_2 satisfies (2.1) and q_1 satisfies (3.1). For $f, \Phi, \Psi \in \mathcal{A}$, let*

$$\left(\frac{z (f * \Phi)' (z)}{(f * \Psi) (z)}\right)^{\gamma} \in \mathcal{H}[1, 1] \cap Q$$

and $\Omega_3(f, \Phi, \Psi, \rho, \delta, \eta, \mu, \gamma; z)$ as defined by (2.9) be univalent in $\mathbb{U}$. If

$$\rho + \delta q_1(z) + \frac{\eta}{q_1(z)} + \mu \frac{z q_1'(z)}{q_1(z)} \prec \Omega_3(f, \Phi, \Psi, \rho, \delta, \eta, \mu, \gamma; z) \prec \rho + \delta q_2(z) + \frac{\eta}{q_2(z)} + \mu \frac{z q_2'(z)}{q_2(z)},$$

then

$$q_1(z) \prec \left(\frac{z (f * \Phi)' (z)}{(f * \Psi) (z)}\right)^{\gamma} \prec q_2(z)$$

and q_1, q_2 are respectively the best subordinant and the best dominant.

By making use of Corollaries 2.3 and 3.3, we obtain the following corollary:

Corollary 4.3. *Let q_1 and q_2 be convex univalent in $\mathbb{U}$ with $q_1(0) = q_2(0) = 1$, $\rho, \delta, \eta, \mu, \gamma \in \mathbb{C}$ such that $\gamma \neq 0$ and let q_2 satisfies (2.1) and q_1 satisfies (3.1). For $f \in \mathcal{A}$, let*

$$\left(1 + \frac{z^{2-\lambda} f''(z)}{(zf'(z))^{1-\lambda}}\right)^{\gamma} \in \mathcal{H}[1,1] \cap Q$$

and $\Omega_5(f, \rho, \delta, \eta, \mu, \gamma, \lambda; z)$ as defined by (2.15) be univalent in $\mathbb{U}$. If

$$\rho + \delta q_1(z) + \frac{\eta}{q_1(z)} + \mu \frac{zq_1'(z)}{q_1(z)} \prec \Omega_5(f, \rho, \delta, \eta, \mu, \gamma, \lambda; z) \prec \rho + \delta q_2(z) + \frac{\eta}{q_2(z)} + \mu \frac{zq_2'(z)}{q_2(z)},$$

then

$$q_1(z) \prec \left(1 + \frac{z^{2-\lambda} f''(z)}{(zf'(z))^{1-\lambda}}\right)^{\gamma} \prec q_2(z)$$

and q_1, q_2 are respectively the best subordinant and the best dominant.

By making use of Corollaries 2.4 and 3.4, we obtain the following corollary:

Corollary 4.4. *Let q_1 and q_2 be convex univalent in $\mathbb{U}$ with $q_1(0) = q_2(0) = 1$, $\rho, \delta, \eta, \mu, \gamma \in \mathbb{C}$ such that $\gamma \neq 0$ and let q_2 satisfies (2.1) and q_1 satisfies (3.1). For $f, \Phi, \Psi \in \mathcal{A}$, let*

$$\left(1 + \frac{z(f * \Phi)''(z)}{(f * \Psi)'(z)}\right)^{\gamma} \in \mathcal{H}[1,1] \cap Q$$

and $\Omega_6(f, \Phi, \Psi, \rho, \delta, \eta, \mu, \gamma; z)$ as defined by (2.17) be univalent in $\mathbb{U}$. If

$$\rho + \delta q_1(z) + \frac{\eta}{q_1(z)} + \mu \frac{zq_1'(z)}{q_1(z)} \prec \Omega_6(f, \Phi, \Psi, \rho, \delta, \eta, \mu, \gamma; z) \prec \rho + \delta q_2(z) + \frac{\eta}{q_2(z)} + \mu \frac{zq_2'(z)}{q_2(z)},$$

then

$$q_1(z) \prec\prec \left(1 + \frac{z(f * \Phi)''(z)}{(f * \Psi)'(z)}\right)^{\gamma} \prec q_2(z)$$

and q_1, q_2 are respectively the best subordinant and the best dominant.

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