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ON REFINEMENT OF JENSEN'S INEQUALITY FOR 3-CONVEX FUNCTION AT A POINT

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Abstract. In this paper, the refinement of Jensen's inequality for convex function given in [5] used to establish the inequalities for classes of 3-convex function at a point. Some new improvements of these new inequalities are also given.

1. Introduction and Preliminary Results

Convex function played a vital role in optimization. Also many important inequalities are due to convexity of the function. One of them is Jensen's inequality because the notion of convex function widely use as classical Jensen's inequality and refinement of Jensen's inequality and it has remain source of valuable results in the literature for many decades. In [8, p. 43] discrete version of Jensen's inequality is given as follows:
Let $f : I \to \mathbb{R}$, where I be an interval in $\mathbb{R}$, is convex, for $n \geq 2$ suppose $(x_1, \ldots, x_n) \in I^n$ and $(p_1, \ldots, p_n)$ is a positive n-tuple and for $k = 1, \ldots, n$ let $P_k := \sum_{i=1}^{k} p_i$, then

$$f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right) \leq \frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i).$$

For refinements and interpolations of Jensen's inequality for the class of convex functions, we refer [4–7] and references there in.

Divided difference is a helpful tool when we are dealing with the functions that have different degrees of smoothness. In [8, p. 14] the divided difference is given as follows.

Definition 1.1. Let g be real valued function defined on $[\alpha, \beta]$. For $r + 1$ distinct points $u_0, u_1, \ldots, u_r$, the r-th order divided difference is defined recursively by

$$[u_i; g] = g(u_i) \;\; i = 0, 1, \ldots, r,$$

and

$$[u_0, u_1, \ldots, u_r; g] = \frac{[u_1, u_2, \ldots, u_r; g] - [u_0, u_1, \ldots, u_{r-1}; g]}{u_r - u_0}.$$

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This is equivalent to

$$[u_0, u_1, \ldots, u_r; g] = \sum_{j=0}^{r} \frac{g(x_j)}{w'(x_j)},$$

where

$$w(x) = \prod_{j=0}^{r} (x - x_j).$$

The r-convex function is characterized by the rth-order divided difference as follows (see [8, p. 14]).

Definition 1.2. A function $g : [\alpha, \beta] \to \mathbb{R}$ is called r-convex function ($r \geq 0$) on $[\alpha, \beta]$ if and only if

$$[u_0, u_1, \ldots, u_r; g] \geq 0 \tag{1.1}$$

for all $(r+1)$ distinct choices on $[\alpha, \beta]$.

If the inequality is reversed then g is n-concave on $[\alpha, \beta]$.

In [5] L. Horváth and J. Pečarić give a refinement of Jensen's inequality for convex function. They define some essential tools to prove the refinement given as follows:
Let X be a set, and:
$P(X)$:=Power set of X,
$|X|$:= Number of elements of X,
$\mathbb{N}$:= Set of natural numbers with 0. Consider $q \geq 1$ and $r \geq 2$ be fixed integers. Define the functions

$$F_{r,s} : \{1, \ldots, q\}^r \to \{1, \ldots, q\}^{r-1} \qquad 1 \leq s \leq r$$

$$F_r : \{1, \ldots, q\}^r \to P\left(\{1, \ldots, q\}^{r-1}\right)$$

and

$$T_r : P\left(\{1, \ldots, q\}^r\right) \to P\left(\{1, \ldots, q\}^{r-1}\right)$$

by

$$F_{r,s}(i_1, \ldots, i_r) := (i_1, i_2, \ldots, i_{s-1}, i_{s+1}, \ldots, i_r) \qquad 1 \leq s \leq r$$

$$F_r(i_1, \ldots, i_r) := \bigcup_{r=1}^{s} \{F_{r,s}(i_1, \ldots, i_r)\}$$

and

$$T_r(I) = \begin{cases} \phi, & I = \phi; \\ \bigcup\limits_{(i_1, \ldots, i_r) \in I} F_r(i_1, \ldots, i_r), & I \neq \phi. \end{cases} \tag{1.2}$$

Next let the function

$$\alpha_{r,i}; \{1, \ldots, q\}^r \to \mathbb{N} \qquad 1 \leq i \leq q \tag{1.3}$$

defined by

$$\alpha_{r,i}(i_1,\dots,i_r) = \text{number of occurences of } i \text{ in the sequence } (i_1,\dots,i_r).$$

For each $I \in P(\{1,\dots,q\}^r)$ let

$$\alpha_{I,i} := \sum_{(i_1,\dots,i_r)\in I} \alpha_{r,i}(i_1,\dots,i_r) \qquad 1 \le i \le r.$$

(H_1) Let n, m be fixed positive integers such that $n \ge 1$, $m \ge 2$ and let I_m be a subset of $\{1,\dots,n\}^m$ such that

$$\alpha_{I_m,i} \ge 1 \qquad 1 \le i \le n.$$

Introduce the sets $I_l \subset \{1,\dots,n\}^l (m-1 \ge l \ge 1)$ inductively by

$$I_{l-1} := T_l(I_l) \qquad m \ge l \ge 2.$$

Obviously, $I_1 = \{1,\dots,n\}$ by (H_1) and this insures that $\alpha_{I_1} = 1 (1 \le i \le n)$. From (1.3) we have $\alpha_{I_l,i} \ge 1 (m-1 \ge l \ge 1, 1 \le i \le n)$. For $m \ge l \ge 2$, and for any $(j_1,\dots,j_{l-1}) \in I_{l-1}$. Let

$$\mathcal{H}_{I_l}(j_1,\dots,j_{l-1}) := \{((i_1,\dots,i_l),k) \times \{1,\dots,l\} | F_{l,k}(i_1,\dots,i_l) = (j_1,\dots,j_{l-1})\}$$

With the help of these sets they defined the function $\eta_{I_m,l} : I_l \to \mathbb{N}(m \ge l \ge 1)$ by

$$\eta_{I_m,m}(i_1,\dots,i_m) := 1 \qquad (i_1,\dots,i_m) \in I_m;$$

$$\eta_{I_m,l-1}(j_1,\dots,j_{l-1}) := \sum_{((i_1,\dots,i_l),k)\in\mathcal{H}_{I_l}(j_1,\dots,j_{l-1})} \eta_{I_m,l}(i_1,\dots,i_l).$$

And they define some special expressions as follows for $m \le l \le 1$, as follows

$$\mathcal{A}_{m,l}(I_m,\mathbf{x},\mathbf{p},f) = \mathcal{A}_{m,l}(I_m,x_1,\dots,x_n,p_1,\dots,p_n;f) := \frac{(m-1)!}{(l-1)!} \sum_{(i_1,\dots,i_l)\in I_l} \eta_{I_m,l}(i_1,\dots,i_l)$$
$$\left(\sum_{j=1}^{l} \frac{p_{i_j}}{\alpha_{I_m,i_j}}\right) f\left(\frac{\sum_{j=1}^{n} \frac{p_{i_j}}{\alpha_{I_m,i_j}} x_{i_j}}{\sum_{j=1}^{l} \frac{p_{i_j}}{\alpha_{I_m,i_j}}}\right) \tag{1.4}$$

Theorem 1.1. *Let $f : C \to \mathbb{R}$ be a convex function where C be a convex subset of real vector space X. Let $p_1, p_2, \dots, p_n$ are positive real numbers such that $\sum_{i=1}^{n} p_i = 1$, then*

$$f\left(\sum_{s=1}^{n} p_s x_s\right) \le \mathcal{A}_{m,m} \le \mathcal{A}_{m,m-1} \le \dots \le \mathcal{A}_{m,2} \le \mathcal{A}_{m,1} = \sum_{s=1}^{n} p_s f(x_s). \tag{1.5}$$

In [2], I. A. Baloch *et al.* introduced the new classes of functions that are $\mathcal{K}_1^a(I)$ and $\mathcal{K}_2^a(I)$ given in the following definition.

Definition 1.3. Let $f : I \to \mathbb{R}$ and $a \in I^\circ$ (I° denote the interior of I). Consider the classes

$$\mathcal{K}_1^a(I) := \Big\{ f : \text{their exist a real number } B \text{ such that} f(x) - \frac{B}{2}x^2 \text{is concave on } I \cap (-\infty, a] \text{and convex on} I \cap [a, \infty) \} \tag{1.6}$$

and

$$\mathcal{K}_2^a(I) := \Big\{ f : \text{their exist a real number } B \text{ such that} f(x) - \frac{B}{2}x^2 \text{is convex on} \\ I \cap (-\infty, a] \text{and concave on} I \cap [a, \infty) \Big\}. \tag{1.7}$$

The function $f \in \mathcal{K}_1^a(I)$ is called 3-convex function at a point and if $f \in \mathcal{K}_2^a(I))$ is 3-concave function at a point.

They also show that the $\mathcal{K}_1^a(I)(\mathcal{K}_2^a(I))$ is larger class of function than the class of all 3-convex(3-concave) functions in the following result (see [2], Theorem 2.4).

Theorem 1.2. *If $g \in \mathcal{K}_1^a(I)(g \in \mathcal{K}_2^a(I))$ for every $a \in I$, then g is 3-convex (3-concave).*

It was also noted that the converse of Theorem 1.2 is not valid in general. For instance $t^4 \in \mathcal{K}_2^a(-1,3)$ but t^4 is not 3-convex at $(-1,3)$.
The Levinson inequality was generalized by replacing with weaker assumption by A. Mercer *et al.* [10]. After that A. Witkowski *et al.* [11] gave further weaker assumption than Mercer to prove Levinson inequality. Then I. A. Baloch *et al.* [2] generalized the result of Mercer and Witkowski by defining a larger class of function that is $\mathcal{K}_1^a(I)$ and $\mathcal{K}_2^a(I)$. After that S. I. Butt *et al.* [12] generalized Popoviciu inequality for $\mathcal{K}_1^a(I)$ and $\mathcal{K}_2^a(I)$ classes. These work motivates us and give an idea to generalize refinement of Jensen's inequality for $\mathcal{K}_1^a(I)$ and $\mathcal{K}_2^a(I)$ classes.
M. Adeel *et al.* [1] later generalized the Levinson inequality for higher order convex function. Butt *et al.* [3] gave some applications to information theory by finding some new bounds for Shannon, relative and Mandelbrot entropies by using discrete and cyclic refinement of Jensen's inequality, and similar type of application to information theory can be find in [9].

2. Main Results

In this section, we use the refinement of Jensen's inequality for convex function given in (1.5) and establish the inequalities for classes of functions $\mathcal{K}_1^a(I)$ and $\mathcal{K}_2^a(I)$ instead of convex function f. We also improve these inequalities.

For this first we define the functional by the differences of refinement of Jensen's inequality given in (1.5) as follows:

$$\Theta_1(f) = \mathcal{A}_{m,r} - f\left(\sum_{s=1}^{n} p_s x_s\right), \qquad r = 1, \ldots, m, \tag{2.1}$$

$$\Theta_2(f) = \mathcal{A}_{m,r} - \mathcal{A}_{m,k}, \qquad 1 \le r < k \le m. \tag{2.2}$$

Remark 2.1. Under the assumption of Theorem 1.1, we have

$$\Theta_i(f) \ge 0, \qquad i = 1, 2. \tag{2.3}$$

And the inequalities (2.3) are reversed if f is concave on C.

Note. In the rest of paper we consider $r < k$.

Theorem 2.1. *Assume* (H_1)*, let* $I = [\alpha, \beta]$ *be an interval. Consider* $\boldsymbol{x} = (x_1, \ldots, x_s) \in [\alpha, \beta]^s$ *and* $\boldsymbol{y} = (y_1, \ldots, y_s) \in [\alpha, \beta]^s$*. Also let there exists* $a \in I$ *such that*

$$\max_i x_i \leq a \leq \min_j y_j.$$

Suppose $\boldsymbol{p} = (p_1, \ldots, p_s) \in (0, \infty)^s$*,* $\boldsymbol{q} = (q_1, \ldots, q_s) \in (0, \infty)^s$ *such that* $\sum\limits_{j=1}^{s} p_j = \sum\limits_{i=1}^{s} q_i = 1$ *and*

$$\mathcal{A}_{m,r}(I_m, \boldsymbol{x}, \boldsymbol{p}, id^2) - \mathcal{A}_{m,k}(I_m, \boldsymbol{x}, \boldsymbol{p}, id^2) = \mathcal{A}_{m,r}(I_m, \boldsymbol{y}, \boldsymbol{q}, id^2) - \mathcal{A}_{m,k}(I_m, \boldsymbol{y}, \boldsymbol{q}, id^2). \tag{2.4}$$

If $f \in \mathcal{K}_1^a(I)$*, then*

$$\mathcal{A}_{m,r}(I_m, \boldsymbol{x}, \boldsymbol{p}, f) - \mathcal{A}_{m,k}(\boldsymbol{x}, \boldsymbol{p}, f) \leq \mathcal{A}_{m,r}(\boldsymbol{y}, \boldsymbol{q}, f) - \mathcal{A}_{m,k}(\boldsymbol{y}, \boldsymbol{q}, f) \tag{2.5}$$

holds.

Proof. Since $H_1(x) := f(x) - \frac{B}{2}x^2$ is concave on $I \cap [\alpha, a]$, therefore from Remark 2.1, we have

$$\begin{aligned} 0 &\geq \mathcal{A}_{m,r}(\mathbf{x}, \mathbf{p}, H_1) - \mathcal{A}_{m,k}(\mathbf{x}, \mathbf{p}, H_1) \\ &= \mathcal{A}_{m,r}(\mathbf{x}, \mathbf{p}, f) - \mathcal{A}_{m,k}(\mathbf{x}, \mathbf{p}, f) - \frac{B}{2}\Big[\mathcal{A}_{m,r}(\mathbf{x}, \mathbf{p}, id^2) \\ &- \mathcal{A}_{m,k}(\mathbf{x}, \mathbf{p}, id^2)\Big]. \end{aligned} \tag{2.6}$$

As $H_2(y) := f(y) - \frac{B}{2}y^2$ is convex on $[a, \beta]$, therefore from Remark 2.1, we get

$$\begin{aligned} 0 &\leq \mathcal{A}_{m,r}(\mathbf{y}, \mathbf{p}, H_2) - \mathcal{A}_{m,k}(\mathbf{y}, \mathbf{p}, H_2) \\ &= \mathcal{A}_{m,r}(\mathbf{y}, \mathbf{p}, f) - \mathcal{A}_{m,k}(\mathbf{y}, \mathbf{p}, f) - \frac{B}{2}\Big[\mathcal{A}_{m,r}(\mathbf{y}, \mathbf{p}, id^2) \\ &- \mathcal{A}_{m,k}(\mathbf{y}, \mathbf{p}, id^2)\Big]. \end{aligned} \tag{2.7}$$

From (2.6) and (2.7), we have

$$\begin{aligned} &\mathcal{A}_{m,r}(\mathbf{x}, \mathbf{p}, f) - \mathcal{A}_{m,k}(\mathbf{x}, \mathbf{p}, f) - \frac{B}{2}\Big[\mathcal{A}_{m,r}(\mathbf{x}, \mathbf{p}, id^2) - \mathcal{A}_{m,k}(\mathbf{x}, \mathbf{p}, id^2)\Big] \\ &\leq \mathcal{A}_{m,r}(\mathbf{y}, \mathbf{p}, f) - \mathcal{A}_{m,k}(\mathbf{y}, \mathbf{p}, f) - \frac{B}{2}\Big[\mathcal{A}_{m,r}(\mathbf{y}, \mathbf{p}, id^2) - \mathcal{A}_{m,k}(\mathbf{y}, \mathbf{p}, id^2)\Big]. \end{aligned}$$

Using the assumption (2.4), we get (2.5). □

Corollary 2.1. *Assume* (H_1)*, let* $I = [0, 2a]$ *be an interval,* $\boldsymbol{x} = (x_1, \ldots, x_s) \in [0, a]^s$*,* $\boldsymbol{y} = (y_1, \ldots, y_s) \in [a, 2a]^s$ *and* $\boldsymbol{p} = (p_1, \ldots, p_s)$ *be positive* n*-tuple such that* $\sum\limits_{j=1}^{s} p_j = 1$*, If* $f \in \mathcal{K}_1^a(I)$*, then the inequality* (2.5) *holds for* $n = m$ *and* $\boldsymbol{p} = \boldsymbol{q}$ *and* $x_1 + y_1 = \ldots = x_s + y_s = 2a$*.*

Proof. Note that

$$id^2\left(\frac{\sum_{j=1}^{k} p_{i_j} y_{i_j}}{\sum_{j=1}^{k} p_{i_j}}\right) = id^2\left(\frac{\sum_{j=1}^{k} p_{i_j}(c - x_{i_j})}{\sum_{j=1}^{k} p_{i_j}}\right) \tag{2.8}$$

$$= c^2 - 2c\frac{\sum_{j=1}^{k} p_{i_j} x_{i_j}}{\sum_{j=1}^{k} p_{i_j}} + \left(\frac{\sum_{j=1}^{k} p_{i_j} x_{i_j}}{\sum_{j=1}^{k} p_{i_j}}\right)^2. \tag{2.9}$$

We can observe that

$$\mathcal{A}_{m,r}(\mathbf{x}, \mathbf{p}, id^2) - \mathcal{A}_{m,k}(\mathbf{x}, \mathbf{p}, id^2) = \mathcal{A}_{m,r}(\mathbf{y}, \mathbf{p}, id^2) - \mathcal{A}_{m,k}(\mathbf{y}, \mathbf{p}, id^2).$$

On following the same step of Theorem 2.1, we get (2.5). □

Remark 2.2. Using (2.6) and (2.7) from proof of Theorem 2.1, we have

$$\mathcal{A}_{m,r}(\mathbf{x}, \mathbf{p}, f) - \mathcal{A}_{m,k}(\mathbf{x}, \mathbf{p}, f) \leq \frac{B}{2}\Big[\mathcal{A}_{m,r}(\mathbf{x}, \mathbf{p}, id^2) - \mathcal{A}_{m,k}(\mathbf{x}, \mathbf{p}, id^2)\Big] \tag{2.10}$$

and

$$\frac{B}{2}\Big[\mathcal{A}_{m,r}(\mathbf{y}, \mathbf{p}, id^2) - \mathcal{A}_{m,k}(\mathbf{y}, \mathbf{p}, id^2)\Big] \leq \mathcal{A}_{m,r}(\mathbf{y}, \mathbf{p}, f) - \mathcal{A}_{m,k}(\mathbf{y}, \mathbf{p}, f). \tag{2.11}$$

Using (2.10) and (2.11), we have the refinement of (2.5) given by

$$\mathcal{A}_{m,r}(\mathbf{x}, \mathbf{p}, f) - \mathcal{A}_{m,k}(\mathbf{x}, \mathbf{p}, f) \leq \frac{B}{2}\left[\mathcal{A}_{m,r}(\mathbf{x}, \mathbf{p}, id^2) - \mathcal{A}_{m,k}(\mathbf{x}, \mathbf{p}, id^2)\right]$$
$$\left(= \frac{B}{2}\left[\mathcal{A}_{m,r}(\mathbf{y}, \mathbf{p}, id^2) - \mathcal{A}_{m,k}(\mathbf{y}, \mathbf{p}, id^2)\right]\right) \leq \mathcal{A}_{m,r}(\mathbf{y}, \mathbf{p}, f) - \mathcal{A}_{m,k}(\mathbf{y}, \mathbf{p}, f).$$

The next result is the generalization of Theorem 2.1, with weaker assumptions on (2.4).

Theorem 2.2. *Assume* (H_1), *let* $I = [\alpha, \beta]$ *be an interval,* $\boldsymbol{x} = (x_1, \ldots, x_s) \in [\alpha, \beta]^s$, $\boldsymbol{y} = (y_1, \ldots, y_s) \in [\alpha, \beta]^s$ *with*

$$\max_i x_i \leq \min_j y_j. \tag{2.12}$$

Also let $\boldsymbol{p} = (p_1, \ldots, p_s) \in (0, \infty)^s$, $\boldsymbol{q} = (q_1, \ldots, q_s) \in (0, \infty)^s$ *such that* $\sum_{j=1}^{s} p_j = \sum_{i=1}^{s} q_i = 1$ *and* $f \in \mathcal{K}_1^a(I)$ *for some* $a \in [\max x_i, \min y_j]$. *Then if*

(i):

$$f''_{-}(\max x_i) \geq 0$$

and

$$\mathcal{A}_{m,r}(\boldsymbol{x}, \boldsymbol{p}, id^2) - \mathcal{A}_{m,k}(\boldsymbol{x}, \boldsymbol{p}, id^2) \leq \mathcal{A}_{m,r}(\boldsymbol{y}, \boldsymbol{q}, id^2) - \mathcal{A}_{m,k}(\boldsymbol{y}, \boldsymbol{q}, id^2)$$

(ii):

$$f''_{+}(\min y_j) \leq 0$$

and

$$\mathcal{A}_{m,r}(\boldsymbol{x}, \boldsymbol{p}, id^2) - \mathcal{A}_{m,k}(\boldsymbol{x}, \boldsymbol{p}, id^2) \geq \mathcal{A}_{m,r}(\boldsymbol{y}, \boldsymbol{q}, id^2) - \mathcal{A}_{m,k}(\boldsymbol{y}, \boldsymbol{q}, id^2)$$

(iii): $f''_{-}(\max x_i) < 0 < f''_{+}(\min y_j)$ *and* f *is 3-convex,*

then (2.5) *holds.*

Proof. Since $f \in \mathcal{K}_1^a[\alpha, \beta]$ for some $a \in [\max x_i, \max y_j]$, therefore their exists a constant B such that $H_1(x) := f(x) - \frac{B}{2}x^2$, is concave on $[\alpha, a]$, such that for $x_1, \ldots, x_s \in I \cap [\alpha, a]$, we have

$$0 \geq \mathcal{A}_{m,r}(\mathbf{x}, \mathbf{p}, H_1) - \mathcal{A}_{m,k}(\mathbf{x}, \mathbf{p}, H_1),$$

that is

$$\begin{aligned} 0 \quad &\geq \quad \mathcal{A}_{m,r}(\mathbf{x}, \mathbf{p}, f) - \mathcal{A}_{m,k}(\mathbf{x}, \mathbf{p}, f) - \frac{B}{2}\Big[\mathcal{A}_{m,r}(\mathbf{x}, \mathbf{p}, id^2) \\ &- \quad \mathcal{A}_{m,k}(\mathbf{x}, \mathbf{p}, id^2)\Big]. \end{aligned} \tag{2.13}$$

Also $H_2(y) := f(y) - \frac{B}{2}y^2$ is convex on $[a, \beta]$, for $y_1, \ldots, y_s \in [a, \beta]$, we have

$$0 \leq \mathcal{A}_{m,r}(\mathbf{y}, \mathbf{p}, H_2) - \mathcal{A}_{m,k}(\mathbf{y}, \mathbf{p}, H_2),$$

that is

$$\begin{aligned} 0 \quad &\geq \quad \mathcal{A}_{m,r}(\mathbf{y}, \mathbf{p}, f) - \mathcal{A}_{m,k}(\mathbf{y}, \mathbf{p}, f) - \frac{B}{2}\Big[\mathcal{A}_{m,r}(\mathbf{y}, \mathbf{p}, id^2) \\ &- \quad \mathcal{A}_{m,k}(\mathbf{y}, \mathbf{p}, id^2)\Big]. \end{aligned} \tag{2.14}$$

From (2.13) and (2.14), we have

$$\begin{aligned} &\mathcal{A}_{m,r}(\mathbf{x}, \mathbf{p}, f) - \mathcal{A}_{m,k}(\mathbf{x}, \mathbf{p}, f) - \frac{B}{2}\left[\mathcal{A}_{m,r}(\mathbf{x}, \mathbf{p}, id^2) - \mathcal{A}_{m,k}(\mathbf{x}, \mathbf{p}, id^2)\right] \\ &\leq \mathcal{A}_{m,r}(\mathbf{y}, \mathbf{p}, f) - \mathcal{A}_{m,k}(\mathbf{y}, \mathbf{p}, f) - \frac{B}{2}\left[\mathcal{A}_{m,r}(\mathbf{y}, \mathbf{p}, id^2) - \mathcal{A}_{m,k}(\mathbf{y}, \mathbf{p}, id^2)\right]. \end{aligned}$$

So

$$\begin{aligned} &\frac{B}{2}\left[\mathcal{A}_{m,r}(\mathbf{y}, \mathbf{p}, id^2) - \mathcal{A}_{m,k}(\mathbf{y}, \mathbf{p}, id^2) - \mathcal{A}_{m,r}(\mathbf{x}, \mathbf{p}, id^2) - \mathcal{A}_{m,k}(\mathbf{x}, \mathbf{p}, id^2)\right] \\ &\qquad \leq \mathcal{A}_{m,r}(\mathbf{y}, \mathbf{p}, f) - \mathcal{A}_{m,k}(\mathbf{y}, \mathbf{p}, f) - \mathcal{A}_{m,r}(\mathbf{x}, \mathbf{p}, f) - \mathcal{A}_{m,k}(\mathbf{x}, \mathbf{p}, f). \end{aligned} \tag{2.15}$$

Now due to concavity of H_1 and convexity of H_2 for every distinct point $\tilde{x}_j \in [\alpha, \max x_i]$ and $\tilde{y}_j \in [\min y_i, \beta]$, $j = 1, 2, 3$, we have

$$[\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, f] \leq B \leq [\tilde{y}_1, \tilde{y}_2, \tilde{y}_3, f]. \tag{2.16}$$

Letting $\tilde{x}_j \nearrow \max x_i$ and $\tilde{y}_j \searrow \min y_j$, we get the inequalities if derivatives exists

$$f''_{-}(\max x_i) \leq B \leq f''_{+}(\min y_i). \tag{2.17}$$

Since from assumption (a), $f''(\max x_i) \geq 0$, therefore $B \geq 0$, so using the assumption

$$\left[\mathcal{A}_{m,r}(\mathbf{y}, \mathbf{p}, id^2) - \mathcal{A}_{m,k}(\mathbf{y}, \mathbf{p}, id^2) - \mathcal{A}_{m,r}(\mathbf{x}, \mathbf{p}, id^2) - \mathcal{A}_{m,k}(\mathbf{x}, \mathbf{p}, id^2)\right] \geq 0,$$

the expression

$$\frac{B}{2}\left[\mathcal{A}_{m,r}(\mathbf{y}, \mathbf{p}, id^2) - \mathcal{A}_{m,k}(\mathbf{y}, \mathbf{p}, id^2) - \mathcal{A}_{m,r}(\mathbf{x}, \mathbf{p}, id^2) - \mathcal{A}_{m,k}(\mathbf{x}, \mathbf{p}, id^2)\right]$$

is non-negative and on using it on left side of (2.15) we have the result (2.5). And similarly for assumption (b), the inequality $f''_+(\min y_j) \leq 0$ gives $B \leq 0$, so the expression with assumption of (b) is also non-negative, this gives the result (2.5). Under the assumption of (c), f''_- and f''_+ are both left and right continuous respectively and both are nondecreasing with $f''_- \leq f''_+$, so their exists a point $\tilde{a} \in [\max x_i, \min y_j]$ such that $f \in \mathcal{K}_1^{\tilde{a}}[\alpha, \beta]$ with constant $\tilde{B} = 0$, and thus we have the inequality (2.5). ☐

Remark 2.3. From the proof of Theorem 2.2, we have

$$\mathcal{A}_{m,r}(\mathbf{x}, \mathbf{p}, f) - \mathcal{A}_{m,k}(\mathbf{x}, \mathbf{p}, f) \leq \frac{B}{2}\left[\mathcal{A}_{m,r}(\mathbf{x}, \mathbf{p}, id^2) - \mathcal{A}_{m,k}(\mathbf{x}, \mathbf{p}, id^2)\right]$$

and

$$\mathcal{A}_{m,r}(\mathbf{y}, \mathbf{p}, f) - \mathcal{A}_{m,k}(\mathbf{y}, \mathbf{p}, f) \geq \frac{B}{2}\left[\mathcal{A}_{m,r}(\mathbf{y}, \mathbf{p}, id^2) - \mathcal{A}_{m,k}(\mathbf{y}, \mathbf{p}, id^2)\right].$$

In Theorem 2.2, B is positive, negative and zero for the assumptions (a), (b) and (c) respectively as discussed in proof. Therefore, we have the better improvement of (2.5) than (2.12) given as

$$\begin{aligned}\mathcal{A}_{m,r}(\mathbf{x}, \mathbf{p}, f) - \mathcal{A}_{m,k}(\mathbf{x}, \mathbf{p}, f) &\leq \frac{B}{2}\left[\mathcal{A}_{m,r}(\mathbf{x}, \mathbf{p}, id^2) - \mathcal{A}_{m,k}(\mathbf{x}, \mathbf{p}, id^2)\right]\\ \leq \frac{B}{2}\left[\mathcal{A}_{m,r}(\mathbf{y}, \mathbf{p}, id^2) - \mathcal{A}_{m,k}(\mathbf{y}, \mathbf{p}, id^2)\right] &\leq \mathcal{A}_{m,r}(\mathbf{y}, \mathbf{p}, f) - \mathcal{A}_{m,k}(\mathbf{y}, \mathbf{p}, f).\end{aligned}$$

If the assumptions of Theorem 2.1 with $f \in \mathcal{K}_2^a[\alpha, \beta]$, the reverse of inequality (2.5) holds. The generalization of this result is proven in the following result.

Theorem 2.3. *Assume* (H_1)*, let* $I = [\alpha, \beta] \subset \mathbb{R}$ *be an interval,* $\boldsymbol{x} = (x_1, \ldots, x_s) \in [\alpha, \beta]^s$*,* $\boldsymbol{y} = (y_1, \ldots, y_s) \in [\alpha, \beta]^s$ *with*

$$\max_i x_i \leq \min_j y_j. \tag{2.18}$$

Also let $\boldsymbol{p} = (p_1, \ldots, p_s) \in (0, \infty)^s$*,* $\boldsymbol{q} = (q_1, \ldots, q_s) \in (0, \infty)^s$ *such that* $\sum_{j=1}^{s} p_j = 1 = \sum_{i=1}^{s} q_i$ *and* $f \in \mathcal{K}_2^a(I)$ *for some* $a \in [\max x_i, \min y_j]$*. Then if*

(i):

$$f''_-(\max x_i) \leq 0$$

and

$$\mathcal{A}_{m,r}(\boldsymbol{x}, \boldsymbol{p}, id^2) - \mathcal{A}_{m,k}(\boldsymbol{x}, \boldsymbol{p}, id^2) \leq \mathcal{A}_{m,r}(\boldsymbol{y}, \boldsymbol{q}, id^2) - \mathcal{A}_{m,k}(\boldsymbol{y}, \boldsymbol{q}, id^2)$$

(ii):

$$f''_{+}(\min y_j) \geq 0$$

and

$$\mathcal{A}_{m,r}(\boldsymbol{x}, \boldsymbol{p}, id^2) - \mathcal{A}_{m,k}(\boldsymbol{x}, \boldsymbol{p}, id^2) \geq \mathcal{A}_{m,r}(\boldsymbol{y}, \boldsymbol{q}, id^2) - \mathcal{A}_{m,k}(\boldsymbol{y}, \boldsymbol{q}, id^2)$$

(iii): $f''_{-}(\max x_i) < 0 < f''_{+}(\min y_j)$ *and* f *is 3-concave,*

then the inequality

$$\mathcal{A}_{m,r}(\boldsymbol{x}, \boldsymbol{p}, f) - \mathcal{A}_{m,k}(\boldsymbol{x}, \boldsymbol{p}, f) \geq \mathcal{A}_{m,r}(\boldsymbol{y}, \boldsymbol{q}, f) - \mathcal{A}_{m,k}(\boldsymbol{y}, \boldsymbol{q}, f) \tag{2.19}$$

holds.

Proof. Since $f \in \mathcal{K}_2^a[\alpha, \beta]$ for some $a \in [\max x_i, \max y_j]$, therefore their exists a constant B such that $H_1(x) = f(x) - \frac{B}{2}x^2$, is convex on $I \cap (-\infty, a]$, such that for $x_1, \ldots, x_s \in I \cap (-\infty, a]$, we have

$$0 \leq \mathcal{A}_{m,r}(\mathbf{x}, \mathbf{p}, H_1) - \mathcal{A}_{m,k}(\mathbf{x}, \mathbf{p}, H_1),$$

that is

$$\begin{aligned} 0 \quad &\leq \quad \mathcal{A}_{m,r}(\mathbf{x}, \mathbf{p}, \ f) - \mathcal{A}_{m,k}(\mathbf{x}, \mathbf{p}, f) - \frac{C}{2}\Big[\mathcal{A}_{m,r}(\mathbf{x}, \mathbf{p}, id^2) \\ &- \quad \mathcal{A}_{m,k}(\mathbf{x}, \mathbf{p}, id^2)\Big]. \end{aligned} \tag{2.20}$$

Also $H_2(y) = f(y) - \frac{B}{2}y^2$ is concave on $I \cap [a, \infty)$, for $y_1, \ldots, y_s \in [a, \infty)$, we have

$$0 \geq \mathcal{A}_{m,r}(\mathbf{y}, \mathbf{p}, H_2) - \mathcal{A}_{m,k}(\mathbf{y}, \mathbf{p}, H_2),$$

that is

$$\begin{aligned} 0 \quad &\geq \quad \mathcal{A}_{m,r}(\mathbf{y}, \mathbf{p}, f) - \mathcal{A}_{m,k}(\mathbf{y}, \mathbf{p}, f) - \frac{B}{2}\Big[\mathcal{A}_{m,r}(\mathbf{y}, \mathbf{p}, id^2) \\ &- \quad \mathcal{A}_{m,k}(\mathbf{y}, \mathbf{p}, id^2)\Big]. \end{aligned} \tag{2.21}$$

From (2.20) and (2.21), we have

$$\begin{aligned} &\mathcal{A}_{m,r}(\mathbf{x}, \mathbf{p}, f) - \mathcal{A}_{m,k}(\mathbf{x}, \mathbf{p}, f) - \frac{B}{2}\left[\mathcal{A}_{m,r}(\mathbf{x}, \mathbf{p}, id^2) - \mathcal{A}_{m,k}(\mathbf{x}, \mathbf{p}, id^2)\right] \\ \geq\ &\mathcal{A}_{m,r}(\mathbf{y}, \mathbf{p}, f) - \mathcal{A}_{m,k}(\mathbf{y}, \mathbf{p}, f) - \frac{B}{2}\left[\mathcal{A}_{m,r}(\mathbf{y}, \mathbf{p}, id^2) - \mathcal{A}_{m,k}(\mathbf{y}, \mathbf{p}, id^2)\right]. \end{aligned}$$

So

$$\begin{aligned} &\frac{B}{2}\left[\mathcal{A}_{m,r}(\mathbf{y}, \mathbf{p}, id^2) - \mathcal{A}_{m,k}(\mathbf{y}, \mathbf{p}, id^2) - \mathcal{A}_{m,r}(\mathbf{x}, \mathbf{p}, id^2) - \mathcal{A}_{m,k}(\mathbf{x}, \mathbf{p}, id^2)\right] \\ &\qquad \geq \mathcal{A}_{m,r}(\mathbf{y}, \mathbf{p}, f) - \mathcal{A}_{m,k}(\mathbf{y}, \mathbf{p}, f) - \mathcal{A}_{m,r}(\mathbf{x}, \mathbf{p}, f) - \mathcal{A}_{m,k}(\mathbf{x}, \mathbf{p}, f). \end{aligned} \tag{2.22}$$

Now due to convexity of H_1 and concavity of H_2 for every distinct point $\tilde{x}_j \in [\alpha, \max x_i]$ and $\tilde{y}_j \in [\min y_i, \beta]$, $j = 1, 2, 3$, we have

$$[\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, f] \geq B \geq [\tilde{y}_1, \tilde{y}_2, \tilde{y}_3, f]. \tag{2.23}$$

Letting $\tilde{x}_j \nearrow \max x_i$ and $\tilde{y}_j \searrow \min y_j$, we get the inequalities if derivatives exists

$$f''_-(\max x_i) \geq B \geq f''_+(\min y_i). \tag{2.24}$$

Since from assumption (a), $f''(\max x_i) \leq 0$, therefore $B \geq 0$, so using the assumption

$$\Big[\mathcal{A}_{m,r}(\mathbf{y}, \mathbf{p}, id^2) - \mathcal{A}_{m,k}(\mathbf{y}, \mathbf{p}, id^2) - \mathcal{A}_{m,r}(\mathbf{x}, \mathbf{p}, id^2) - \mathcal{A}_{m,k}(\mathbf{x}, \mathbf{p}, id^2)\Big] \geq 0$$

we have

$$\frac{B}{2}\Big[\mathcal{A}_{m,r}(\mathbf{y}, \mathbf{p}, id^2) - \mathcal{A}_{m,k}(\mathbf{y}, \mathbf{p}, id^2) - \mathcal{A}_{m,r}(\mathbf{x}, \mathbf{p}, id^2) - \mathcal{A}_{m,k}(\mathbf{x}, \mathbf{p}, id^2)\Big]$$

is negative and on using it on left side of (2.22) we have the result (2.5). And similarly for assumption (b), the inequality $f''_+(\min y_j) \geq 0$ gives $B > 0$, so the expression with assumption of (b) is also positive, this gives the result (2.5). Under the assumption of (c), f''_- and f''_+ are left and right continuous respectively and both are decreasing with $f''_- \geq f''_+$, so their exists a point $\tilde{a} \in [\max x_i, \min y_j]$ such that $f \in \mathcal{K}_1^{\tilde{a}}[\alpha, \beta]$ with constant $\tilde{B} = 0$, and thus we have the inequality (2.19). □

Remark 2.4. In Theorem 2.3, B is negative or positive or zero under the assumption $(i), (ii)$ and (iii) respectively as discussed earlier in the proof of the Theorem 2.3. Therefore we get the improvement of (2.19) as follows.

$$\begin{aligned}
&\mathcal{A}_{m,r}(\mathbf{x}, \mathbf{p}, f) - \mathcal{A}_{m,k}(\mathbf{x}, \mathbf{p}, f) \geq \mathcal{A}_{m,r}(\mathbf{x}, \mathbf{p}, id^2) - \mathcal{A}_{m,k}(\mathbf{x}, \mathbf{p}, id^2) \\
&\geq \mathcal{A}_{m,r}(\mathbf{y}, \mathbf{q}, id^2) - \mathcal{A}_{m,k}(\mathbf{y}, \mathbf{q}, id^2) \geq \mathcal{A}_{m,r}(\mathbf{y}, \mathbf{q}, f) - \mathcal{A}_{m,k}(\mathbf{y}, \mathbf{q}, f).
\end{aligned}$$

Remark 2.5. Theorem 2.1, Remark 2.2, Theorem 2.2, Remark 2.3 and Theorem 2.3 are also valid for the differences given in (2.1) and (2.2) for $r = 1, \ldots, m$ and $1 \leq r < k \leq m$ respectively.

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Competing interests

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Ethical approval

This article does not contain any studies with human participants or animals performed by any of the authors.

Authors contribution

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