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ANALYSIS OF THE NUTRIENT-PHYTOPLANKTON-ZOOPLANKTON SYSTEM WITH NON-LOCAL AND NON-SINGULAR KERNEL

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Abstract. The system of nutrient-phytoplankton-zooplankton (NPZ) is analyzed. This system is extended to the ABC fractional derivative. Then, with the help of Banach fixed point theorem, an existence solution is made. Finally, the uniqueness solution of the system is analyzed with the help of Sumudu transform method.

1. Introduction

Especially with the introduction of new kernels in the last decade, fractional calculus has become a perfect tool for analyzing mathematical models. Fractional calculus is ideal for identifying the hereditary characteristics of memory, various materials and operations. However, the analysis made with discrete derivatives neglect these situations. This can be considered one of the most important advantages of fractional calculus. Proposed by [1], the local and non-singular kernel produces good results in the analysis of mathematical models. Numerical results have been obtained in the literature by analyzing many mathematical models and systems with the help of the AB kernel [2–8]. There are also important studies in the literature regarding the application of inequalities to fractional integral operators[9–11].

In this paper, the system of nutrient-phytoplankton-zooplankton (NPZ) has been analyzed. Some studies have been conducted using this system. Edward [12] investigated two population dynamics models in order to study the sensitivity to different parameter values and complexity. Legendre and Rassoulzadegan [13] described the trophic pathways continuity between systems dominated by systems and herbivorous food network dominated by microbial loop. They suggested that continuity shift from herbivorous network to a "multivariable goods network," to the network, and finally to the microbial cycle. Megrey et al. [14] put forth dynamically link a population dynamics model based on fish bioenergetics to the NPZ model. Zhang and Wang [15] examined the NPZ model in an aquatic environment.

Key words and phrases. Nutrient-phytoplankton-zooplankton model, ABC fractional operator, Fixed-point theorem, Sumudu transform.

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Javidi and Ahmad [16] studied the Routh-Hurwitz criteria were applied to discuss stability analysis in terms of reproduction numbers (related to ecological and disease phenomena) of biologically appropriate equilibria for the given system. Ghanbari and GÃşmez-Aguilar [17] investigated the NPZ model, which includes different types of fractional derivative operators. In particular, they modeled the interaction of NP and its predatory zooplankton. In this study, we will investigate the behaviour of nutrient-phytoplankton-zooplankton system by using Atangana-Baleanu-Caputo fractional operator. First, we will demonstrate the existence of a solution for the nutrient-phytoplankton-zooplankton system in the ABC derivative using uniqueness and theorem which includes Picard's fixed-point theorem.

The rest of this study is organized as follows. In the second chapter, Atangana-Baleanu-Caputo (ABC) fractional derivative and Atangana-Baleanu (AB) integral operator is introduced and necessary theorems are given. In the third section, the existence solution of the recovered system is made. In the fourth section, the uniqueness solution was obtained with the help of Sumudu transform. Lastly, the fifth section is reserved for the conclusion.

2. Preliminaries

In second section, fundamental definitions and theorems will be given which are related to Atangana-Baleanu-Caputo fractional derivative and integral operators.

Definition 2.1. The well-known fractional order Caputo derivative is defined as follows [18],

$$ {}_{a}^{C}D_{t}^{\nu}g(t) = \frac{1}{\Gamma(m-\nu)} \int_{a}^{t} \frac{g^{(m)}(\omega)}{(t-\omega)^{\nu+1-m}} d\omega, \quad m-1 < \nu < m \in N. $$

with $g \in H^1(a,b), b > a$.

Definition 2.2. The Riemann-Liouville fractional integral is defined as [19]:

$$ J^{\nu}g(t) = \frac{1}{\Gamma(\nu)} \int_{a}^{t} g(\omega)(t-\omega)^{\nu-1} d\omega. $$

Definition 2.3. The Sobolev space of order 1 in (a,b) is defined as [20]:

$$ H^1(a,b) = \{u \in L^2(a,b) : u' \in L^2(a,b)\}. $$

Definition 2.4. Let a function $g \in H^1(a,b)$ and $\nu \in (0,1)$. The AB fractional derivative in Caputo sense of order ν of g with a based point a is defined as [1]:

$$ {}_{a}^{\mathrm{ABC}}D_{t}^{\nu}g(t) = \frac{B(\nu)}{1-\nu} \int_{a}^{t} g'(\omega) E_{\nu}\Big[-\frac{\nu}{1-\nu}(t-\omega)^{\nu} \Big] d\omega, $$

where $B(\nu)$ has the same properties as in Caputo-Fabrizio case [21], and is defined as

$$B(\nu) = 1 - \nu + \frac{\nu}{\Gamma(\nu)},$$

$E_{\nu,\beta}(\lambda^\nu)$ is the Mittag-Leffler function, defined in terms of a series as the following entire function

$$E_{\nu,\beta}(z) = \sum_{k=0}^{\infty} \frac{(\lambda^\nu)^k}{\Gamma(\nu k + \beta)}, \quad \nu > 0, \quad \lambda < \infty, \quad \beta > 0, \quad \lambda = -\nu(1-\nu)^{-1}.$$

Definition 2.5. Let a function $g \in H^1(a,b)$ and $\nu \in (0,1)$. The AB fractional derivative in Riemann-Liouville sense of order ν of g with a based point a is defined as [1]:

$${}_{a}^{ABR}D_t^\nu g(t) = \frac{B(\nu)}{1-\nu}\frac{d}{dt}\int_a^t g(\omega) E_\nu\Big[-\frac{\nu}{1-\nu}(t-\omega)^\nu\Big]d\omega,$$

when the function g is constant, we get zero.

Definition 2.6. The Atangana-Baleanu fractional integral of order ν with base point a is defined as [1]:

$${}^{AB}I_t^\nu g(t) = \frac{1-\nu}{B(\nu)} g(t) + \frac{\nu}{B(\nu)\Gamma(\nu)} \int_a^t g(\omega)(t-\omega)^{\nu-1} d\omega.$$

Theorem 2.1. *The following time fractional ordinary differential equation*

$${}_{a}^{ABR}D_t^\nu g(t) = u(t) - u(0)$$

has a unique solution which takes the inverse Laplace transform and uses the convolution theorem below [8]*,*

$$g(t) - g(0) = \frac{1-\nu}{B(\nu)} u(t) + \frac{\nu}{B(\nu)\Gamma(\nu)} \int_a^t u(\omega)(t-\omega)^{\nu-1} d\omega.$$

3. Existence of Solution for The Nutrient-Phytoplankton-Zooplankton (NPZ) System

The nutrient-phytoplankton-zooplankton system with the classical integer order is as follows:

$$\begin{aligned}
\frac{dK(t)}{dt} &= a_0 - a_1K(t) - a_2K(t)L(t) + a_3L(t) + a_4M(t)\\
\frac{dL(t)}{dt} &= a_5K(t)L(t) - a_6L(t) - \frac{a_7L(t)M(t)}{a_8 + L(t)}\\
\frac{dM(t)}{dt} &= \frac{a_9L(t) + M(t)}{a_8 + L(t)} - a_{10}L(t)M(t) - a_{11}M(t).
\end{aligned} \tag{3.1}$$

where $a_i, (i = 0, 1, 2, \dots, 11)$ are positive contants. In the system (3.1), $K(t)$ represents the concentration of nutrient, $L(t)$ represents the biomass of phytoplankton which also produces toxicant harmful to the zooplankton biomass, $M(t)$ represents the concentration of zooplankton population.

When the system (3.1) is extended to ABC fractional derivative, the following system of equations is obtained.:

$$\begin{aligned} {}_a^{ABC}D_t^{\nu}K(t) &= a_0 - a_1K(t) - a_2K(t)L(t) + a_3L(t) + a_4M(t) \\ {}_a^{ABC}D_t^{\nu}L(t) &= a_5K(t)L(t) - a_6L(t) - \frac{a_7L(t)M(t)}{a_8 + L(t)} \\ {}_a^{ABC}D_t^{\nu}M(t) &= \frac{a_9L(t) + M(t)}{a_8 + L(t)} - a_{10}L(t)M(t) - a_{11}M(t). \end{aligned} \tag{3.2}$$

where $\nu \in (0, 1)$ is the order of the fractional derivative. Then the following initial values,

$$K_0 = K_0(t) > 0, \quad L_0 = L_0(t) > 0, \quad M_0 = M_0(t) > 0.$$

Theorem 3.1. *The ordinary differential equation as below,*

$${}_a^{ABC}D_t^{\nu}f(t) = u(t),$$

has a unique solution using the convolution theorem with taking the inverse Laplace transform and below [6],

$$f(t) = \frac{1-\nu}{B(\nu)}u(t) + \frac{\nu}{B(\nu)\Gamma(\nu)}\int_a^t u(t)(t-\omega)^{\nu-1}d\omega$$

According to the theorem (3.1), the system (3.2) can be written as below,

$$\begin{aligned} K(t) - K_0(t) &= \frac{1-\nu}{B(\nu)}(a_0 - a_1K(t) - a_2K(t)L(t) + a_3L(t) + a_4M(t)) \\ &+ \frac{\nu}{B(\nu)\Gamma(\nu)}\int_0^t (t-\tau)^{\nu-1}(a_0 - a_1K(\tau) - a_2K(\tau)L(\tau) + a_3L(\tau) + a_4M(\tau))d\tau \\ L(t) - L_0(t) &= \frac{1-\nu}{B(\nu)}(a_5K(t)L(t) - a_6L(t) - \frac{a_7L(t)M(t)}{a_8 + L(t)}) \\ &+ \frac{\nu}{B(\nu)\Gamma(\nu)}\int_0^t (t-\tau)^{\nu-1}(a_5K(\tau)L(\tau) - a_6L(\tau) - \frac{a_7L(\tau)M(\tau)}{a_8 + L(\tau)})d\tau \\ M(t) - M_0(t) &= \frac{1-\nu}{B(\nu)}(\frac{a_9L(t) + M(t)}{a_8 + L(t)} - a_{10}L(t)M(t) - a_{11}M(t)) \\ &+ \frac{\nu}{B(\nu)\Gamma(\nu)}\int_0^t (t-\tau)^{\nu-1}(\frac{a_9L(\tau) + M(\tau)}{a_8 + L(\tau)} - a_{10}L(\tau)M(\tau) - a_{11}M(\tau))d\tau. \end{aligned} \tag{3.3}$$

The above system (3.3) can be iteratively represent as,

$$K_0(t) = f_1(t)$$
$$L_0(t) = f_2(t)$$
$$M_0(t) = f_3(t)$$

$$\begin{aligned}
K_{n+1}(t) &= \frac{1-\nu}{B(\nu)}(a_0 - a_1K_n(t) - a_2K_n(t)L_n(t) + a_3L_n(t) + a_4M_n(t)) \\
&+ \frac{\nu}{B(\nu)\Gamma(\nu)}\int_0^t (t-\tau)^{\nu-1})(a_0 - a_1K_n(\tau) - a_2K_n(\tau)L_n(\tau) + a_3L_n(\tau) + a_4M_n(\tau))d\tau \\
L_{n+1}(t) &= \frac{1-\nu}{B(\nu)}(a_5K_n(t)L_n(t) - a_6L_n(t) - \frac{a_7L_n(t)M_n(t)}{a_8 + L_n(t)}) \\
&+ \frac{\nu}{B(\nu)\Gamma(\nu)}\int_0^t (t-\tau)^{\nu-1}(a_5K_n(\tau)L_n(\tau) - a_6L_n(\tau) - \frac{a_7L_n(\tau)M_n(\tau)}{a_8 + L_n(\tau)})d\tau \\
M_{n+1}(t) &= \frac{1-\nu}{B(\nu)}(\frac{a_9L_n(t) + M_n(t)}{a_8 + L_n(t)} - a_{10}L_n(t)M_n(t) - a_{11}M_n(t)) \\
&+ \frac{\nu}{B(\nu)\Gamma(\nu)}\int_0^t (t-\tau)^{\nu-1}(\frac{a_9L_n(\tau) + M_n(\tau)}{a_8 + L_n(\tau)} - a_{10}L_n(\tau)M_n(\tau) - a_{11}M_n(\tau))d\tau.
\end{aligned} \tag{3.4}$$

Since the number of series terms is tends to infinity, the following system of equations is obtained with the iterative formula of the Picard series if it is taken the limit greater than n, we hope to get the full solution of the equation as follows:

$$\lim_{n\to\infty} K_n(t) = K(t)$$
$$\lim_{n\to\infty} L_n(t) = L(t)$$
$$\lim_{n\to\infty} M_n(t) = M(t)$$

Let us to illustrate the existence of solution with the following operator.

$$g_1(t,x) = a_0 - a_1K(t) - a_2K(t)L(t) + a_3L(t) + a_4M(t)$$
$$g_2(t,x) = a_5K(t)L(t) - a_6L(t) - \frac{a_7L(t)M(t)}{a_8 + L(t)}$$
$$g_3(t,x) = \frac{a_9L(t) + M(t)}{a_8 + L(t)} - a_{10}L(t)M(t) - a_{11}M(t).$$

Let us examine,

$$N_1 = \sup_{C_{a,b_1}} ||g_1(t,x)|| \quad N_2 = \sup_{C_{a,b_2}} ||g_1(t,,)|| \quad N_3 = \sup_{C_{a,b_3}} ||g_1(t,z)||,$$

where

$$C_{a,b_1} = [t-a, t+a] \times [x-b_1, x+b_1] = A_1 \times B_1$$
$$C_{a,b_2} = [t-a, t+a] \times [x-b_2, x+b_2] = A_1 \times B_2$$
$$C_{a,b_3} = [t-a, t+a] \times [x-b_3, x+b_3] = A_1 \times B_3.$$

Now we will using the metric on $C[b, c_i], (i = 1, 2, 3)$ with the Banach fixed-point theorem made by the norm,

$$||X(t)||_\infty = \sup_{t \in [t-b, t+b]} |f(t)|.$$

PicardâĂŹs operator as follows:

$$O : C(A_1, B_1, B_2, B_3) \to C(A_1, B_1, B_2, B_3).$$

For simplicity, let us $g_i(a, t) = X(t), g_i(a, 0) = X_0(t), (i = 1, 2, 3)$. Then the system is reduced the following,

$$OX(t) = X_0(t) + G(t, X(t))\frac{1-\nu}{B(\nu)} + \frac{\nu}{B(\nu)\Gamma(\nu)}\int_0^t (t-\tau)^{\nu-1} G(\tau, X(\tau))d\tau$$

where X is the matrice of given as,

$$X(t) = \begin{cases} K(t) \\ L(t) \\ M(t) \end{cases} \qquad X_0(t) = \begin{cases} K_0(t) \\ L_0(t) \\ M_0(t) \end{cases} \qquad G(t, X(t)) = \begin{cases} g_1(t, x) \\ g_2(t, x) \\ g_3(t, x) \end{cases}$$

Therefore,

$$\begin{aligned} ||OX(t) - X_0(t) &= \left|\left| G(t, X(t))\frac{1-\nu}{B(\nu)} + \frac{\nu}{B(\nu)\Gamma(\nu)}\int_0^t (t-\tau)^{\nu-1} G(\tau, X(\tau))d\tau \right|\right| \\ &\leq \frac{1-\nu}{B(\nu)}||G(t, X(t))|| + \frac{\nu}{B(\nu)\Gamma(\nu)}\int_0^t (t-\tau)^{\nu-1}||G(\tau, X(\tau))||d\tau \\ &\leq \frac{1-\nu}{B(\nu)}N \\ &= max\{N_1, N_2, N_3\} + \frac{\nu}{B(\nu)}Na^\nu \leq aN \leq b = max\{b_1, b_2, b_3\}. \end{aligned}$$

It can be written above inequality,

$$a < \frac{b}{N}$$

Considering the equality below,

$$||OX_1 - OX_2||_\infty = \sup_{t \in A} |X_1 - X_2|.$$

With the help of the defined operator, the following inequality is achieved.

$$
\begin{aligned}
||OX_1 - OX_2|| &= \Bigg|\Bigg|\frac{1-\nu}{B(\nu)}\Big(G(t,X_1(t)) - G(t,X_2(t))\Big) \\
&\quad + \frac{\nu}{B(\nu)\Gamma(\nu)}\int_0^t (t-\tau)^{\nu-1}\Big(G(\tau,X_1(\tau)) - G(\tau,X_2(\tau))\Big)d\tau\Bigg|\Bigg| \\
&\leq \frac{1-\nu}{B(\nu)}||G(t,X_1(t)) - G(t,X_2(t))|| \\
&\quad + \frac{\nu}{B(\nu)\Gamma(\nu)}\int_0^t (t-\tau)^{\nu-1}||G(\tau,X_1(\tau)) - G(\tau,X_2(\tau))||d\tau \\
&\leq \frac{1-\nu}{B(\nu)}Q||X_1(t) - X_2(t)||d\tau \\
&\quad + \frac{\nu Q}{B(\nu)\Gamma(\nu)}\int_0^t (t-\tau)^{\nu-1}||X_1(\tau) - X_2(\tau)||d\tau \\
&\leq \left\{\frac{1-\nu}{B(\nu)}Q\right\}||X_1(t) - X_2(t)|| \leq aQ||X_1(t) - X_2(t)||,
\end{aligned}
$$

where $Q < 1$. Since G is a contraction we obtained the value of $aQ < 1$, so the defined operator O is a contraction too. This shows that the system is a unique set of solutions.

4. Uniqueness of Solution for The Nutrient-Phytoplankton-Zooplankton (NPZ) System

The extended nutrient-phytoplankton-zooplankton system to the ABC fractional derivative is difficult to solve by analytical methods. Therefore, an iterative method is needed. The modified transformation-based method and the iterative method will be used here to obtain a particular set of solutions for the model. Having the capability of maintaining the parity of the function, the Sumudu transformation operator will be used here. We will start to investigate the uniqueness of solutions of the model with the theorem given below.

Theorem 4.1. *Let $f \in H^1(a,b), b > a, \nu \in (0,1)$, the Sumudu transform of ABC fractional derivative is given as* [6],

$$
ST\{{}_a^{ABC}D_t^{\nu} f(t)\} = \frac{B(\nu)}{1-\nu}\Big(\nu\Gamma(\nu+1)E_\nu\big(-\frac{1}{1-\nu}p^\nu\big)\Big)(ST(f(t) - f(0)) \tag{4.1}
$$

Now, we will solve the system (3.2) with the Sumudu transform with both sides. Then,

$$
\begin{aligned}
&\frac{B(\nu)}{1-\nu}\Big(\nu\Gamma(\nu+1)E_{\nu}\big(-\frac{1}{1-\nu}p^{\nu}\big)\Big)\big(ST(K(t))-K(0)\big)\\
&=ST\{a_0-a_1K(t)-a_2K(t)L(t)+a_3L(t)+a_4M(t)\}\\
&\frac{B(\nu)}{1-\nu}\Big(\nu\Gamma(\nu+1)E_{\nu}\big(-\frac{1}{1-\nu}p^{\nu}\big)\Big)\big(ST(L(t))-L(0)\big)\\
&=ST\Big\{a_5K(t)L(t)-a_6L(t)-\frac{a_7L(t)M(t)}{a_8+L(t)}\Big\}\\
&\frac{B(\nu)}{1-\nu}\Big(\nu\Gamma(\nu+1)E_{\nu}\big(-\frac{1}{1-\nu}p^{\nu}\big)\Big)\big(ST(L(t))-L(0)\big)\\
&=ST\Big\{\frac{a_9L(t)+M(t)}{a_8+L(t)}-a_{10}L(t)M(t)-a_{11}M(t)\Big\}.
\end{aligned}
$$

When the above system is rearranged for $\lambda=-\frac{1}{1-\nu}$, the following system is obtained.

$$
\begin{aligned}
&ST(K(t))=K(0)+\frac{1-\nu}{B(\nu)(\nu\Gamma(\nu+1)E_{\nu}(\lambda p^{\nu}))}\times\\
&ST\{a_0-a_1K(t)-a_2K(t)L(t)+a_3L(t)+a_4M(t)\}\\
&ST(L(t))=L(0)+\frac{1-\nu}{B(\nu)(\nu\Gamma(\nu+1)E_{\nu}(\lambda p^{\nu}))}\times\\
&ST\Big\{a_5K(t)L(t)-a_6L(t)-\frac{a_7L(t)M(t)}{a_8+L(t)}\Big\}\\
&ST(M(t))=M(0)+\frac{1-\nu}{B(\nu)(\nu\Gamma(\nu+1)E_{\nu}(\lambda p^{\nu}))}\times\\
&ST\Big\{\frac{a_9L(t)+M(t)}{a_8+L(t)}-a_{10}L(t)M(t)-a_{11}M(t)\Big\}
\end{aligned}
$$

Then we get the following recursive formula,

$$\begin{aligned}
&K_{n+1}(t) = K_n(0) + ST^{-1}\left\{\frac{1-\nu}{B(\nu)(\nu\Gamma(\nu+1)E_\nu(\lambda p^\nu))}\times\right.\\
&\left.ST\{a_0 - a_1K(t) - a_2K(t)L(t) + a_3L(t) + a_4M(t)\}\right\}\\
&L_{n+1}(t) = L_n(0) + ST^{-1}\left\{\frac{1-\nu}{B(\nu)(\nu\Gamma(\nu+1)E_\nu(\lambda p^\nu))}\times\right.\\
&\left.ST\left\{a_5K(t)L(t) - a_6L(t) - \frac{a_7L(t)M(t)}{a_8+L(t)}\right\}\right\}\\
&M_{n+1}(t) = M_n(0) + ST^{-1}\left\{\frac{1-\nu}{B(\nu)(\nu\Gamma(\nu+1)E_\nu(\lambda p^\nu))}\times\right.\\
&\left.ST\left\{\frac{a_9L(t)+M(t)}{a_8+L(t)} - a_{10}L(t)M(t) - a_{11}M(t)\right\}\right\}.
\end{aligned} \tag{4.2}$$

Finally the solution of system (4.2) is provided by,

$$\begin{aligned}
K(t) &= \lim_{n\to\infty} K_n(t)\\
L(t) &= \lim_{n\to\infty} L_n(t)\\
M(t) &= \lim_{n\to\infty} M_n(t)
\end{aligned}$$

Theorem 4.2. *Let* $(x, ||.||)$ *be a Banach space and* H *a self-map of* X *satisfying*

$$||H_x - H_y|| \leq C||x - H_x|| + c||x - y||$$

For all x, y *in* X *where* $0 \leq C, 0 \leq c \leq 1$. *Suppose that* H *is Picard's* $H-$ *stable [15].*

For all $x, y \in X$ where $0 \leq C, 0 \leq c \leq 1$. Suppose that H is PicardâĂŹs H-stable [22]. Then, it is considered that the recursive formula system (4.2) with system (3.3)

$$\begin{aligned}
&K_{n+1}(t) = K_n(0) + ST^{-1}\{\theta ST\{a_0 - a_1K(t) - a_2K(t)L(t) + a_3L(t) + a_4M(t)\}\}\\
&L_{n+1}(t) = L_n(0) + ST^{-1}\left\{\theta ST\left\{a_5K(t)L(t) - a_6L(t) - \frac{a_7L(t)M(t)}{a_8+L(t)}\right\}\right\}\\
&M_{n+1}(t) = M_n(0) + ST^{-1}\left\{\theta ST\left\{\frac{a_9L(t)+M(t)}{a_8+L(t)} - a_{10}L(t)M(t) - a_{11}M(t)\right\}\right\}.
\end{aligned} \tag{4.3}$$

where $\theta = \frac{1-\nu}{B(\nu)(\nu\Gamma(\nu+1)E_\nu(\lambda p^\nu))}$ is the fractional Lagrange multiplier.

Theorem 4.3. *Let* H *bea self-map defined as*

$$\begin{aligned}
H(K_n(t)) =& K_{n+1}(t) = K_n(t) \\
&+ ST^{-1}\{\theta ST\{a_0 - a_1K(t) - a_2K(t)L(t) + a_3L(t) + a_4M(t)\}\} \\
H(L_n(t)) =& L_{n+1}(t) = L_n(t) \\
&+ ST^{-1}\left\{\theta ST\left\{a_5K(t)L(t) - a_6L(t) - \frac{a_7L(t)M(t)}{a_8 + L(t)}\right\}\right\} \\
H(M_n(t)) =& M_{n+1}(t) = M_n(t) \\
&+ ST^{-1}\left\{\theta ST\left\{\frac{a_9L(t) + M(t)}{a_8 + L(t)} - a_{10}L(t)M(t) - a_{11}M(t)\right\}\right\}.
\end{aligned}$$

is $H-$ stable in $L^1(a,b)$ if

$$\begin{aligned}
&||H(K_n(t) - K_m(t))|| \leq K_{n+1}(t) = ||K_n(t) - K_m(t)||(1 - (a_1 + a_2)k(\varphi)) \\
&||H(L_n(t) - L_m(t))|| \leq L_{n+1}(t) = ||L_n(t) - L_m(t)||(1 + a_5l(\varphi)) \\
&||H(M_n(t) - M_m(t))|| \leq M_{n+1}(t) = ||M_n(t) - M_m(t)||(1 + (1 - a_{10} - a_{11})m(\varphi)).
\end{aligned} \tag{4.4}$$

Proof. Firstly we need to show that H has a fixed point. To goal this, we need to evaluate the following for all $(n,m) \in N \times N$

$$\begin{aligned}
H(L_n(t)) - H(L_m(t)) =& L_n(t) - L_m(t) \\
&+ ST^{-1}\left\{\theta ST\left\{a_5K_n(t)L_n(t) - a_6L_n(t) - \frac{a_7L_n(t)M_n(t)}{a_8 + L_n(t)}\right\}\right\} \\
&- ST^{-1}\left\{\theta ST\left\{a_5K_m(t)L_m(t) - a_6L_m(t) - \frac{a_7L_m(t)M_m(t)}{a_8 + L_m(t)}\right\}\right\}
\end{aligned} \tag{4.5}$$

If the norm is taken on both sides of the equation of (4.5), we get

$$\begin{aligned}
||H(L_n(t)) - H(L_m(t))|| =& \Bigg|\Bigg| L_n(t) - L_m(t) \\
&+ ST^{-1}\left\{\theta ST\left\{a_5K_n(t)L_n(t) - a_6L_n(t) - \frac{a_7L_n(t)M_n(t)}{a_8 + L_n(t)}\right\}\right\} \\
&- ST^{-1}\left\{\theta ST\left\{a_5K_m(t)L_m(t) - a_6L_m(t) - \frac{a_7L_m(t)M_m(t)}{a_8 + L_m(t)}\right\}\right\}\Bigg|\Bigg| \\
&\leq ||L_n(t) - L_m(t)|| + ||ST^{-1}\{\theta ST\{A_n(t)L_n(t) - A_m(t)L_m(t)\}\}
\end{aligned}$$

where

$$A_n(t) = a_5 K_n(t) - a_6 - \frac{a_7 M_n(t)}{a_8}$$
$$A_m(t) = a_5 K_m(t) - a_6 - \frac{a_7 M_m(t)}{a_8}$$

Therefore,

$$||H(L_n(t)) - H(L_m(t)) \leq ||L_n(t) - L_m(t)||(1 + a_5 l(\nu))$$

where $l(\nu)$ is the $ST^{-1}\{\theta.ST\}$. Similarly, the following inequalities are obtained,

$$||H(K_n(t) - K_m(t))|| \leq K_{n+1}(t) = ||K_n(t) - K_m(t)||(1 - (a_1 + a_2)k(\varphi))$$
$$||H(M_n(t) - M_m(t))|| \leq M_{n+1}(t) = ||M_n(t) - M_m(t)||(1 + (1 - a_{10} - a_{11})m(\varphi)).$$

This completes the proof. □

5. Conclusion

In this study, we expanded the nutrient-phytoplankton-zooplankton model to the concept of ABC fractional derivative operator and AB fractional integral opetator. Using fixed point theorem, we examined the existence of a generalized model. Also with the sumudu transform, we presented the derivation of the model and confirmed the stability analysis of the method with the H-stable approach.

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