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CERTAIN RESULTS ON RUSCHEWEYH OPERATOR

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ABSTRACT. In the present paper, we find certain results on Ruscheweyh operator using differential subordination. We derive certain results for starlike, convex and close-to-convex functions as particular cases to our main result.

1. INTRODUCTION

Let $\mathcal{H}$ denote the class of functions f, analytic in the open unit disk $\mathbb{E} = \{z \in \mathbb{C} : |z| < 1\}$ in the complex plane $\mathbb{C}$. Let $\mathcal{A}_n$ be the subclass of $\mathcal{H}$, consisting functions f of the form

$$f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k, \text{ for } n \in \mathbb{N} = \{1,\ 2,\ 3,\ \ldots\},$$

in $\mathbb{E}$. A function $f \in \mathcal{A}_n$ is said to be starlike of order α if and only if

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha,\ 0 \leq \alpha < 1,\ z \in \mathbb{E}.$$

The class of such functions is denoted by $\mathcal{S}^*_n(\alpha)$. A function $f \in \mathcal{A}_n$ is said to be convex of order α in $\mathbb{E}$, if and only if

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha,\ 0 \leq \alpha < 1,\ z \in \mathbb{E}.$$

Let $\mathcal{K}_n(\alpha)$, denote the class of all functions $f \in \mathcal{A}_n$ that are convex of order α in $\mathbb{E}$.
A function $f \in \mathcal{A}_n$ is said to be in the class $\mathcal{C}(\alpha)$ of close-to-convex of order α in $\mathbb{E}$ if it satisfies

$$\Re\left(\frac{zf'(z)}{g(z)}\right) > \alpha,\ z \in \mathbb{E};\ 0 \leq \alpha < 1 \text{ and where } g \in \mathcal{S}^*_n.$$

Note that $\mathcal{S}^*_1(\alpha) = \mathcal{S}^*(\alpha),\ \mathcal{K}_1(\alpha) = \mathcal{K}(\alpha)$ and $\mathcal{C}_1(\alpha) = \mathcal{C}(\alpha),\ 0 \leq \alpha < 1$ are the usual classes of univalent starlike, convex and close-to-convex of order α respectively. Also note that $\mathcal{A}_1 = \mathcal{A}$.
Let f and g be two analytic functions in open unit disk $\mathbb{E}$. Then we say f is subordinate to g in $\mathbb{E}$, denoted by $f \prec g$, if there exist a Schwarz function w analytic in $\mathbb{E}$, with $w(0) = 0$

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and $|w(z)| < 1,\ z \in \mathbb{E}$ such that $f(z) = g(w(z)),\ z \in \mathbb{E}$. In case the function g is univalent, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(\mathbb{E}) \subset g(\mathbb{E})$.
The Taylor's series expansions of $f,\ g \in \mathcal{A}$ are given as

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \text{ and } g(z) = z + \sum_{k=2}^{\infty} b_k z^k.$$

Then the Hadamard product or convolution of f and g is denoted by $f * g$ and defined as

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k.$$

For $f \in \mathcal{A}$, Ruscheweyh [5] defined

$$R^{\lambda} f(z) = \frac{z}{(1-z)^{\lambda+1}} * f(z),\ \lambda \geq -1,\ z \in \mathbb{E}. \tag{1.1}$$

And for $\lambda = n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ *where* $\mathbb{N} = \{1,\ 2,\ 3, \dots\}$, he observed that

$$R^n f(z) = \frac{z(z^{n-1} f(z))^{(n)}}{n!},\ z \in \mathbb{E}.$$

This symbol $R^{\lambda},\ \lambda \in \mathbb{N}_0$ was named as Ruscheweyh derivative of f of order λ by Al-Amiri [3]. Lecko et al. [1] observed that for $\lambda \geq -1$, the expression given in (1.1) becomes

$$R^{\lambda} f(z) = z + \sum_{k=2}^{\infty} \frac{(\lambda+1)(\lambda+2)\dots(\lambda+k-1)}{(k-1)!} a_k z^k,\ z \in \mathbb{E},$$

and for every $\lambda > -1$

$$\begin{aligned} R^1 R^{\lambda} f(z) = z(R^{\lambda} f)'(z) &= z\left(\frac{z}{(1-z)^{\lambda+1}} * f(z)\right)' \\ &= \frac{z}{(1-z)^{\lambda+1}} * (zf'(z)) = R^{\lambda}(zf'(z)) = R^{\lambda} R^1 f(z),\ z \in \mathbb{E}. \end{aligned} \tag{1.2}$$

We notice that

$$R^{-1} f(z) = z,\ R^0 f(z) = f(z),\ R^1 f(z) = zf'(z) \text{ and } R^2 f(z) = zf'(z) + \frac{z^2}{2} f''(z),$$

and so on. For $\lambda \in \mathbb{N}_0$ and for $z \in \mathbb{E}$, we have

$$z(R^{\lambda} f)'(z) = (\lambda+1) R^{\lambda+1} f(z) - \lambda R^{\lambda} f(z). \tag{1.3}$$

Note that, this identity also holds for $\lambda = -1$.
In 2006, Wang et al. [7] studied the class $Q(\alpha,\ \beta,\ \gamma)$ defined as:

$$Q(\alpha,\ \beta,\ \gamma) = \{f \in \mathcal{A} : \Re[\alpha(f(z)/z) + \beta f'(z)] > \gamma,\ (\alpha,\ \beta) > 0,\ 0 \leq \gamma < \alpha + \beta \leq 1;\ z \in \mathbb{E}\}.$$

They provided the extreme points and radius of univalence for the members of this class. Then in 2007, Gao et al. [2] studied the following subclass of $\mathcal{A}$:

$$R(\beta,\ \alpha) = \{f \in \mathcal{A} : \Re(f'(z) + \alpha z f''(z)) > \beta,\ z \in \mathbb{E}\},$$

where $\beta < 1,\ \alpha > 0$. They determined the extreme points of $R(\beta,\ \alpha)$ and obtain sharp bounds for $\Re(f'(z))$ and $\Re(f(z)/z)$. They also determined the number $\beta(\alpha)$ such that

$R(\beta,\ \alpha) \subset \mathcal{S}^*$, for certain fixed number α in $[1,\ \infty)$. Recently, Shams et al. [6] studied the Ruscheweyh derivative operator for $f \in \mathcal{A}_n$ that satisfies the inequality given below:

$$\left| \left(1-\alpha+\alpha(\lambda+2)\frac{R^{\lambda+2}f(z)}{R^{\lambda+1}f(z)}\right)\left(\frac{R^{\lambda+1}f(z)}{R^{\lambda}f(z)}\right)^{\mu} - \alpha(\lambda+1)\left(\frac{R^{\lambda+1}f(z)}{R^{\lambda}f(z)}\right)^{\mu+1} - 1\right| < M,$$

and obtained the values of $M,\ \alpha,\ \delta$ and μ for which the function had become starlike of order δ. In the present paper, we study the following operator for $f \in \mathcal{A}$ given below as:

$$\frac{R^{\lambda+1}f(z)}{R^{\lambda}f(z)}\left[1-\alpha+\alpha\left((\lambda+2)\frac{R^{\lambda+2}f(z)}{R^{\lambda+1}f(z)} - (\lambda+1)\frac{R^{\lambda+1}f(z)}{R^{\lambda}f(z)}\right)\right],$$

where α is a non-zero complex number and $\lambda \in \mathbb{N}_1 = \mathbb{N} \cup \{0,\ -1\}$, where $\mathbb{N} = \{1,\ 2,\ 3, \dots\}$, and obtain certain conditions for starlikeness, convexity and close-to-convexity.

2. Preliminary

To prove our main result, we shall make use of the following lemma of Miller Mocanu [4].

Lemma 2.1. *Let $F(z) = 1 + b_1 z + b_2 z^2 + \dots$ be analytic in $\mathbb{E}$ and $h(z)$ be analytic and convex function in $\mathbb{E}$ with $h(0) = 1$. If*

$$F(z) + \frac{1}{c} zF'(z) \prec h(z), \tag{2.1}$$

where $c \neq 0$ and $\Re(c) > 0$, then

$$F(z) \prec cz^{-c}\int_0^z t^{c-1}h(t)\,dt,$$

and $cz^{-c}\int_0^z t^{c-1}h(t)\,dt$ is the best dominant of the differential subordination (2.1).

3. Main Results

Theorem 3.1. *Let α be a non-zero complex number such that $\Re(\alpha) > 0$ and h be analytic and convex in $\mathbb{E}$ with $h(0) = 1$. If $f \in \mathcal{A}$ satisfies*

$$\frac{R^{\lambda+1}f(z)}{R^{\lambda}f(z)}\left[1-\alpha+\alpha\left((\lambda+2)\frac{R^{\lambda+2}f(z)}{R^{\lambda+1}f(z)} - (\lambda+1)\frac{R^{\lambda+1}f(z)}{R^{\lambda}f(z)}\right)\right] \prec h(z),$$

then

$$\frac{R^{\lambda+1}f(z)}{R^{\lambda}f(z)} \prec \frac{1}{\alpha}\int_0^1 t^{\frac{1}{\alpha}-1}h(zt)\,dt,\ \ \lambda \in \mathbb{N}_1,\ \ z \in \mathbb{E}.$$

Proof. Define $u(z) = \dfrac{R^{\lambda+1}f(z)}{R^{\lambda}f(z)},\ z \in \mathbb{E}$.
On differentiating logarithmically, we get

$$\frac{zu'(z)}{u(z)} = \frac{z(R^{\lambda+1}f(z))'}{R^{\lambda+1}f(z)} - \frac{z(R^{\lambda}f(z))'}{R^{\lambda}f(z)}.$$

Using the equality (1.3), the above equation reduces to,

$$u(z) + \alpha zu'(z) = \frac{R^{\lambda+1}f(z)}{R^{\lambda}f(z)}\left[1-\alpha+\alpha\left((\lambda+2)\frac{R^{\lambda+2}f(z)}{R^{\lambda+1}f(z)} - (\lambda+1)\frac{R^{\lambda+1}f(z)}{R^{\lambda}f(z)}\right)\right]. \tag{3.1}$$

Taking $c = \dfrac{1}{\alpha}$ and using Lemma 2.1, from (3.1), we get

$$\frac{R^{\lambda+1}f(z)}{R^{\lambda}f(z)}\left[1-\alpha+\alpha\left((\lambda+2)\frac{R^{\lambda+2}f(z)}{R^{\lambda+1}f(z)}-(\lambda+1)\frac{R^{\lambda+1}f(z)}{R^{\lambda}f(z)}\right)\right]\prec h(z),$$

then

$$\frac{R^{\lambda+1}f(z)}{R^{\lambda}f(z)}\prec\frac{1}{\alpha}z^{-\frac{1}{c}}\int_0^z t^{\frac{1}{\alpha}-1}h(t)\,dt=\frac{1}{\alpha}\int_0^1 t^{\frac{1}{\alpha}-1}h(zt)\,dt.$$

□

On selecting $h(z) = \dfrac{1+2z(\alpha-\beta-\alpha\beta)+(2\beta-1)z^2}{(1-z)^2}$ in Theorem 3.1, where $0 \le \beta < 1$ and α is same as given in this theorem, we obtain the following result:

Corollary 3.1. *Let α be a non-zero complex number such that $\Re(\alpha) > 0$ and let $f \in \mathcal{A}$ satisfies*

$$\frac{R^{\lambda+1}f(z)}{R^{\lambda}f(z)}\left[1-\alpha+\alpha\left((\lambda+2)\frac{R^{\lambda+2}f(z)}{R^{\lambda+1}f(z)}-(\lambda+1)\frac{R^{\lambda+1}f(z)}{R^{\lambda}f(z)}\right)\right]\prec\frac{1+2z(\alpha-\beta-\alpha\beta)+(2\beta-1)z^2}{(1-z)^2}$$

then

$$\frac{R^{\lambda+1}f(z)}{R^{\lambda}f(z)}\prec\frac{1+(1-2\beta)z}{1-z},\ 0\le\beta<1,\ \lambda\in\mathbb{N}_1,\ z\in\mathbb{E}.$$

Taking the dominant $h(z) = 1 + (1 + \alpha)az,\ 0 < a \le 1$ and α is same as in Theorem 3.1, we have the following result from this theorem:

Corollary 3.2. *Let α be a non-zero complex number such that $\Re(\alpha) > 0$ and let $f \in \mathcal{A}$ satisfy*

$$\left|\frac{R^{\lambda+1}f(z)}{R^{\lambda}f(z)}\left[1-\alpha+\alpha\left((\lambda+2)\frac{R^{\lambda+2}f(z)}{R^{\lambda+1}f(z)}-(\lambda+1)\frac{R^{\lambda+1}f(z)}{R^{\lambda}f(z)}\right)\right]-1\right|<(1+\alpha)a$$

then

$$\left|\frac{R^{\lambda+1}f(z)}{R^{\lambda}f(z)}-1\right|<a,\ 0<a\le1,\ \lambda\in\mathbb{N}_1,\ z\in\mathbb{E}.$$

In Theorem 3.1, when $h(z) = 1 + \frac{4}{3}(1+\alpha)z + \frac{2}{3}(1+2\alpha)z^2$ is selected as a dominant, where α is same as in this theorem, we get:

Corollary 3.3. *Let α be a non-zero complex number such that $\Re(\alpha) > 0$ and let $f \in \mathcal{A}$ satisfy*

$$\frac{R^{\lambda+1}f(z)}{R^{\lambda}f(z)}\Big[1-\alpha+\alpha\Big((\lambda+2)\frac{R^{\lambda+2}f(z)}{R^{\lambda+1}f(z)}-(\lambda+1)\frac{R^{\lambda+1}f(z)}{R^{\lambda}f(z)}\Big)\Big]\prec1+\frac{4}{3}(1+\alpha)z+\frac{2}{3}(1+2\alpha)z^2$$

then

$$\frac{R^{\lambda+1}f(z)}{R^{\lambda}f(z)}\prec1+\frac{4}{3}z+\frac{2}{3}z^2,\ \lambda\in\mathbb{N}_1,\ z\in\mathbb{E}.$$

4. Conditions for Starlikeness, Convexity and Close-to-convexity

Setting $\lambda = -1$ in Corollary 3.1, we obtain:

Corollary 4.1. *Let α be a non-zero complex number such that $\Re(\alpha) > 0$. If $f \in \mathcal{A}$ satisfies*
$$(1-\alpha)\frac{f(z)}{z} + \alpha f'(z) \prec \frac{1+2z(\alpha-\beta-\alpha\beta)+(2\beta-1)z^2}{(1-z)^2},$$
then
$$\frac{f(z)}{z} \prec \frac{1+(1-2\beta)z}{1-z},\ \ 0 \le \beta < 1,\ \ z \in \mathbb{E}.$$

For $\lambda = -1$ and replacing $f(z)$ with $zf'(z)$ in Corollary 3.1, we get:

Corollary 4.2. *Let α be a non-zero complex number such that $\Re(\alpha) > 0$. If $f \in \mathcal{A}$ satisfies*
$$f'(z) + \alpha z f''(z) \prec \frac{1+2z(\alpha-\beta-\alpha\beta)+(2\beta-1)z^2}{(1-z)^2},$$
then
$$f'(z) \prec \frac{1+(1-2\beta)z}{1-z},\ \ 0 \le \beta < 1,\ \ z \in \mathbb{E}.$$
Hence $f \in \mathcal{C}(\beta)$.

Selecting $\lambda = 0$ in Corollary 3.1, we obtain:

Corollary 4.3. *Let α be a non-zero complex number such that $\Re(\alpha) > 0$. If $f \in \mathcal{A}$ satisfies*
$$\frac{zf'(z)}{f(z)}\left[1+\alpha\left(1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right)\right] \prec \frac{1+2z(\alpha-\beta-\alpha\beta)+(2\beta-1)z^2}{(1-z)^2},$$
then
$$\frac{zf'(z)}{f(z)} \prec \frac{1+(1-2\beta)z}{1-z},\ \ 0 \le \beta < 1,\ \ z \in \mathbb{E}.$$
i.e. $f \in \mathcal{S}^*(\beta)$.

Putting $\lambda = 0$ and replacing $f(z)$ with $zf'(z)$ in Theorem 3.1, we get the following result:

Corollary 4.4. *Let α be a non-zero complex number such that $\Re(\alpha) > 0$. If $f \in \mathcal{A}$ satisfies*
$$\left(1+\frac{zf''(z)}{f'(z)}\right)\left[1+\alpha\left(\frac{2zf''(z)+z^2f'''(z)}{f'(z)+zf''(z)}-\frac{zf''(z)}{f'(z)}\right)\right] \prec \frac{1+2z(\alpha-\beta-\alpha\beta)+(2\beta-1)z^2}{(1-z)^2},$$
then
$$1+\frac{zf''(z)}{f'(z)} \prec \frac{1+(1-2\beta)z}{1-z},\ \ 0 \le \beta < 1,\ \ z \in \mathbb{E}.$$
i.e. $f \in \mathcal{K}(\beta)$.

Selecting $\lambda = -1$ in Corollary 3.2, we have:

Corollary 4.5. *Let α be a non-zero complex number such that $\Re(\alpha) > 0$ and let $f \in \mathcal{A}$ satisfy*

$$\left|(1-\alpha)\frac{f(z)}{z} + \alpha f'(z) - 1\right| < (1+\alpha)a$$

then

$$\left|\frac{f(z)}{z} - 1\right| < a, \ 0 < a \leq 1, \ z \in \mathbb{E}.$$

For $\lambda = -1$ and replacing $f(z)$ with $zf'(z)$ in Corollary 3.2, we get:

Corollary 4.6. *Let α be a non-zero complex number such that $\Re(\alpha) > 0$. If $f \in \mathcal{A}$ satisfies*

$$|f'(z) + \alpha z f''(z) - 1| < (1+\alpha)a,$$

then

$$|f'(z) - 1| < a, \ 0 < a \leq 1, \ z \in \mathbb{E}.$$

Taking $\lambda = 0$ in Corollary 3.2, we obtain:

Corollary 4.7. *Let α be a non-zero complex number such that $\Re(\alpha) > 0$. If $f \in \mathcal{A}$ satisfies*

$$\left|\frac{zf'(z)}{f(z)}\left[1 + \alpha\left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right)\right] - 1\right| < (1+\alpha)a,$$

then

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < a, \ 0 < a \leq 1, \ z \in \mathbb{E}.$$

Selecting $\lambda = 0$ and on replacing $f(z)$ with $zf'(z)$ in Corollary 3.2, we get the following result:

Corollary 4.8. *Let α be a non-zero complex number such that $\Re(\alpha) > 0$. If $f \in \mathcal{A}$ satisfies*

$$\left|\left(1 + \frac{zf''(z)}{f'(z)}\right)\left[1 + \alpha\left(\frac{2zf''(z) + z^2 f'''(z)}{f'(z) + zf''(z)} - \frac{zf''(z)}{f'(z)}\right)\right] - 1\right| < (1+\alpha)a,$$

then

$$\left|\frac{zf''(z)}{f'(z)}\right| < a, \ 0 < a \leq 1, \ z \in \mathbb{E}.$$

Selecting $\lambda = -1$ in Corollary 3.3, we have:

Corollary 4.9. *Let α be a non-zero complex number such that $\Re(\alpha) > 0$ and let $f \in \mathcal{A}$ satisfy*

$$(1-\alpha)\frac{f(z)}{z} + \alpha f'(z) \prec 1 + \frac{4}{3}(1+\alpha)z + \frac{2}{3}(1+2\alpha)z^2,$$

then

$$\frac{f(z)}{z} \prec 1 + \frac{4}{3}z + \frac{2}{3}z^2, \ z \in \mathbb{E}.$$

For $\lambda = -1$ and replacing $f(z)$ with $zf'(z)$ in Corollary 3.3, we get:

Corollary 4.10. *Let α be a non-zero complex number such that $\Re(\alpha) > 0$. If $f \in \mathcal{A}$ satisfies*

$$f'(z) + \alpha z f''(z) \prec 1 + \frac{4}{3}(1+\alpha)z + \frac{2}{3}(1+2\alpha)z^2,$$

then

$$f'(z) \prec 1 + \frac{4}{3}z + \frac{2}{3}z^2, \ \ z \in \mathbb{E}.$$

Hence $f \in \mathcal{C}$.

Taking $\lambda = 0$ in Corollary 3.3, we obtain:

Corollary 4.11. *Let α be a non-zero complex number such that $\Re(\alpha) > 0$. If $f \in \mathcal{A}$ satisfies*

$$\frac{zf'(z)}{f(z)}\left[1 + \alpha\left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right)\right] \prec 1 + \frac{4}{3}(1+\alpha)z + \frac{2}{3}(1+2\alpha)z^2,$$

then

$$\frac{zf'(z)}{f(z)} \prec 1 + \frac{4}{3}z + \frac{2}{3}z^2, \ \ z \in \mathbb{E}.$$

i.e. $f \in \mathcal{S}^*$.

Selecting $\lambda = 0$ and on replacing $f(z)$ with $zf'(z)$ in Corollary 3.3, we get the following result:

Corollary 4.12. *Let α be a non-zero complex number such that $\Re(\alpha) > 0$. If $f \in \mathcal{A}$ satisfies*

$$\left(1 + \frac{zf''(z)}{f'(z)}\right)\left[1 + \alpha\left(\frac{2zf''(z) + z^2 f'''(z)}{f'(z) + zf''(z)} - \frac{zf''(z)}{f'(z)}\right)\right] \prec 1 + \frac{4}{3}(1+\alpha)z + \frac{2}{3}(1+2\alpha)z^2,$$

then

$$1 + \frac{zf''(z)}{f'(z)} \prec 1 + \frac{4}{3}z + \frac{2}{3}z^2, \ \ z \in \mathbb{E}.$$

i.e. $f \in \mathcal{K}$.

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