Turkish J. Ineq., 4 (1) (2020), Pages 42 –49 .

Turkish Journal of
INEQUALITIES

Available online at www.tjinequality.com

ON HÖLDER-MCCARTHY INEQUALITIES AND APPLICATIONS TO THE POWERS OF SOME OPERATORS IN HILBERT SPACES

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ABSTRACT. In this note, starting with some natural refinements of Hölder's inequalities, we deduce some refinements to the well known Hölder-McCarthy inequalities obtained by C. A. McCarthy in [6]. Using these refinements and the spectral theorem for positive bounded operators, we provide some improvements to certain inequalities obtained by H. Albadawi and K. Shebrawi in [2]. Also, we give some improvements to certain inequalities obtained by S. S. Dragomir in [3].

1. INTRODUCTION

The following result is proved by McCarthy in [6].

Theorem 1.1. *(Hölder McCarthy) Let A be a positive operator on a Hilbert space $\mathcal{H}$. Then, for any $x \in \mathcal{H}$ and a given positive real number r,*

$$\langle A^r x, x\rangle \leq \langle Ax, x\rangle^r ||x||^{2(1-r)} \qquad 0 < r \leq 1, \tag{1.1}$$

$$\langle A^r x, x\rangle \geq \langle Ax, x\rangle^r ||x||^{2(1-r)} \qquad r \geq 1. \tag{1.2}$$

H. Albadawi, and K. Shebrawi [2] proved the following result.

Theorem 1.2. *Let A and B be operators in $\mathcal{B}(\mathcal{H})$, and let α be any positive real number such that $0 < \alpha < 1$. Then for $r \geq 2$*

$$||A+B||^r \leq 2^{r-1}\Big(\alpha|||A|^r + |B|^r|| + (1-\alpha)|||A^*|^r + |B^*|^r||\Big). \tag{1.3}$$

S.S. Dragomir [3] proved the following result.

Theorem 1.3. *For any $A, B \in \mathcal{B}(\mathcal{H})$, any $\alpha \in (0,1)$ and $r \geq 1$, we have the vector inequality:*

$$|\langle Ax, By\rangle|^{2r} \leq \alpha\langle (A^*A)^{\frac{r}{\alpha}} x, x\rangle + (1-\alpha)\langle (B^*B)^{\frac{r}{1-\alpha}} y, y\rangle, \tag{1.4}$$

Key words and phrases. Hölder-McCarthy inequalities, Young's inequality, Hölder inequality, Spectral theorem, positive bounded operators.

2010 *Mathematics Subject Classification.* Primary: 47A30. Secondary: 47A12.

for any $x, y \in \mathcal{H}$ *with* $||x|| = ||y|| = 1$.
In particular, we have the norm inequality

$$||B^*A||^{2r} \leq \alpha||(A^*A)^{\frac{r}{\alpha}}|| + (1-\alpha)||(B^*B)^{\frac{r}{1-\alpha}}||. \tag{1.5}$$

The aim of this note is to establish refinements to the inequality (1.1), (1.2), (1.3), (1.4) and (1.5).

2. Preliminaries

Before giving our results, we need the following lemmas, the first lemma which gives a refinement of Young's inequality.

Lemma 2.1. [1] *Let* a *and* b *two positive numbers. Then for all integer* $n \geq 1$ *and* $\alpha \in]0,1[$ *we have:*

$$a^\alpha b^{1-\alpha} \leq (\alpha a^{\frac{1}{n}} + (1-\alpha)b^{\frac{1}{n}})^n \leq \alpha a + (1-\alpha)b,$$

moreover, if we set $U_n(\alpha, a, b) := (\alpha a^{\frac{1}{n}} + (1-\alpha)b^{\frac{1}{n}})^n$, *then* $U_n(\alpha, a, b)$ *is a sequence decreasing and we have*

$$\lim_{n\to\infty} U_n(\alpha, a, b) = a^\alpha b^{1-\alpha}.$$

As a consequence, we obtain the following result which provides a refinement of Hölder inequality.

Lemma 2.2. [1] *Let* p *and* q *two numbers such that* $\frac{1}{p} + \frac{1}{q} = 1$ *and* $p, q > 1$, *and* f, g *nonnegative* $\mu-$*mesurable functions. Then for all* $n \geq 2$ *we have:*

$$\begin{aligned} \int_\Omega |f(x)g(x)|d\mu(x) &\leq \Big(\frac{1}{p^n} + \frac{1}{q^n}\Big)||f||_p||g||_q \\ &+ \sum_{k=1}^{n-1} \binom{k}{n} \frac{1}{p^k q^{n-k}} ||f||_p^{1-\frac{kp}{n}} ||g||_q^{1-\frac{(n-k)q}{n}} \int_\Omega |f(x)|^{\frac{kp}{n}} |g(x)|^{\frac{(n-k)q}{n}} d\mu(x) \\ &\leq ||f||_p||g||_q. \end{aligned}$$

The third lemma follows from spectral theorem for positive operators and Jensen's inequality, this lemma is proved in [6].

Lemma 2.3. *(McCarthy inequality) Let* $A \in \mathcal{B}(\mathcal{H})$ $A \geq 0$ *and let* $x \in \mathcal{H}$ *be any unit vector. Then*

(a) $\langle Ax, x\rangle^r \leq \langle A^r x, x\rangle$ *for* $r \geq 1$,

(b) $\langle A^r x, x\rangle \leq \langle Ax, x\rangle^r$ *for* $0 < r \leq 1$.

The fourth lemma follows from the spectral theorem for self-adjoint operators and Jensen's inequality, this lemma is proved by B. Mond and J. Pecaric in [7].

Lemma 2.4. *Let A be a selfadjoint operator on the Hilbert space $\mathcal{H}$ and assume that $Sp(A) \subseteq [m, M]$ for some scalars m, M with $m < M$. If f is convex function on $[m, M]$, then*

$$f\Big(\langle Ax, x\rangle\Big) \leq \langle f(A)x, x\rangle,$$

for each $x \in \mathcal{H}$ with $||x|| = 1$.

The fifth lemma is known as mixed Schwarz inequality, this lemma is proved in [8], (see also [5].)

Lemma 2.5. *Let $A \in \mathcal{B}(\mathcal{H})$ and $\alpha \in (0, 1)$. Then*

$$|\langle Ax, y\rangle|^2 \leq \langle |A|^{2\alpha}x, x\rangle\langle |A^*|^{2(1-\alpha)}y, y\rangle \qquad \forall x, y \in \mathcal{H}.$$

The sixth lemma is a simple consequence of the classical Jensen's inequality.

Lemma 2.6. *If a and b are nonegative real numbers, then*

$$(a + b)^r \leq 2^{r-1}(a^r + b^r) \qquad \text{for } r \geq 1.$$

The siventh lemma is an elementary inequality.

Lemma 2.7. *For $a_1, a_2, b_1, b_2 \geq 0$, and $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. We have*

$$a_1b_1 + a_2b_2 \leq \Big(a_1^p + a_2^p\Big)^{1/p}\Big(b_1^q + b_2^q\Big)^{1/q}.$$

3. The Results

The first result of this note aims to provide some refinements to the inequality (1.1) of Theorem 1.1 of McCarthy.

Theorem 3.1. *Let A be a positive operator on a Hilbert space $\mathcal{H}$. Then, for any $x \in \mathcal{H}$ and a given positive real number $r \in (0, 1)$, and $n \geq 2$ we have:*

$$\begin{aligned}
\langle A^r x, x\rangle &\leq \Big(r^n + (1-r)^n\Big)\langle Ax, x\rangle^r ||x||^{2(1-r)} \\
&+ \sum_{k=1}^{n-1} \binom{k}{n} r^k (1-r)^{(n-k)} \langle Ax, x\rangle^{r-\frac{k}{n}} ||x||^{2\left((1-r)-\frac{n-k}{n}\right)} \langle A^{\frac{k}{n}}x, x\rangle \\
&\leq \langle Ax, x\rangle^r ||x||^{2(1-r)}.
\end{aligned}$$

Proof. We apply Lemma 2.2, by setting $\Omega := \sigma(A)$ and considering the measure $d\mu(\lambda) = d\langle E(\lambda)x, x\rangle$, choose $f(\lambda) = \lambda^r$, $g(\lambda) = 1$, and $p = \frac{1}{r}$, $q = \frac{1}{1-r}$ Then we have:

$$\int_\Omega |f(\lambda)g(\lambda)|d\mu(\lambda) = \int_{\sigma(A)} \lambda^r d\langle E(\lambda)x, x\rangle = \langle A^r x, x\rangle$$

$$||f||_p = \Big(\int_\Omega |f(\lambda)|^p d\mu(\lambda)\Big)^{1/p} = \Big(\int_{\sigma(A)} \lambda d\langle E(\lambda)x, x\rangle\Big)^r = \langle Ax, x\rangle^r$$

and

$$||g||_q = \Big(\int_\Omega |g(\lambda)|^q d\mu(\lambda)\Big)^{1/q} = \Big(\int_{\sigma(A)} d\langle E(\lambda)x, x\rangle\Big)^{1-r} = \langle E(\sigma(A))x, x\rangle^{1-r} = ||x||^{2(1-r)}$$

$$||f||_p^{1-\frac{kp}{n}} = \langle Ax, x\rangle^{r-\frac{k}{n}}, \qquad ||g||_q^{1-\frac{(n-k)q}{n}} = ||x||^{2\left((1-r)-\frac{(n-k)}{n}\right)}$$

$$\int_\Omega |f(\lambda)|^{\frac{kp}{n}} |g(\lambda)|^{\frac{(n-k)q}{n}} d\mu(\lambda) = \int_{\sigma(A)} \lambda^{\frac{k}{n}} d\langle E(\lambda)x, x\rangle = \langle A^{\frac{k}{n}}x, x\rangle.$$

Then by Lemma 2.2, we have:

$$\begin{aligned} \langle A^r x, x\rangle &\leq \Big(r^n + (1-r)^n\Big)\langle Ax, x\rangle^r ||x||^{2(1-r)} \\ &+ \sum_{k=1}^{n-1} \binom{k}{n} r^k (1-r)^{(n-k)} \langle Ax, x\rangle^{r-\frac{k}{n}} ||x||^{2\left((1-r)-\frac{n-k}{n}\right)} \langle A^{\frac{k}{n}}x, x\rangle. \end{aligned}$$

By using the inequality (1.1), for all $1 \leq k \leq n-1$, we have

$$\langle A^{\frac{k}{n}}x, x\rangle \leq \langle Ax, x\rangle^{\frac{k}{n}} ||x||^{2(1-\frac{k}{n})} \qquad \Big(\tfrac{k}{n} \in (0,1)\Big).$$

Therefore, we have

$$\begin{aligned} \langle A^r x, x\rangle &\leq \Big(r^n + (1-r)^n\Big)\langle Ax, x\rangle^r ||x||^{2(1-r)} \\ &+ \sum_{k=1}^{n-1} \binom{k}{n} r^k (1-r)^{(n-k)} \langle Ax, x\rangle^{r-\frac{k}{n}} ||x||^{2\left((1-r)-\frac{n-k}{n}\right)} \langle A^{\frac{k}{n}}x, x\rangle \\ &\leq \langle Ax, x\rangle^r ||x||^{2(1-r)}. \end{aligned}$$

This ends the proof. □

The second result of this note aims is a refinements to the inequality (1.2) of Theorem 1.1 of McCarthy.

Theorem 3.2. *Let A be a positive operator on a Hilbert space $\mathcal{H}$. Then, for any $x \in \mathcal{H}$ and a given positive real number $r \geq 1$, and $n \geq 2$ we have:*

$$\begin{aligned} \langle Ax, x\rangle^r ||x||^{2(1-r)} &\leq \Big(\Big(\frac{1}{r}\Big)^n + \Big(\frac{r-1}{r}\Big)^n\Big)\langle A^r x, x\rangle \\ &+ \sum_{k=1}^{n-1} \binom{k}{n} \Big(\frac{1}{r}\Big)^k \Big(\frac{r-1}{r}\Big)^{(n-k)} \langle A^r x, x\rangle^{1-\frac{k}{n}r} ||x||^{-2\left(\frac{n-k}{n}r\right)} \langle A^{\frac{k}{n}r}x, x\rangle^r \\ &\leq \langle A^r x, x\rangle. \end{aligned}$$

Proof. We apply Lemma 2.2, by setting $\Omega := \sigma(A)$ and considering the measure $d\mu(\lambda) = d\langle E(\lambda)x, x\rangle$, choose $f(\lambda) = \lambda$, $g(\lambda) = 1$, and $p = r$, $q = \frac{r}{r-1}$. Then we have

$$\int_\Omega |f(\lambda)g(\lambda)| d\mu(\lambda) = \int_{\sigma(A)} \lambda d\langle E(\lambda)x, x\rangle = \langle Ax, x\rangle$$

$$||f||_p = \Big(\int_\Omega |f(\lambda)|^p d\mu(\lambda)\Big)^{1/p} = \Big(\int_{\sigma(A)} \lambda^r d\langle E(\lambda)x, x\rangle\Big)^{1/r} = \langle A^r x, x\rangle^{1/r}$$

and

$$||g||_q = \Big(\int_\Omega |g(\lambda)|^q d\mu(\lambda)\Big)^{1/q} = \Big(\int_{\sigma(A)} d\langle E(\lambda)x, x\rangle\Big)^{\frac{r-1}{r}} = \langle E(\sigma(A))x, x\rangle^{\frac{r-1}{r}} = ||x||^{2(\frac{r-1}{r})}$$

$$||f||_p^{1-\frac{kp}{n}} = \langle Ax, x\rangle^{\frac{1}{r}-\frac{k}{n}}, \qquad ||g||_q^{1-\frac{(n-k)q}{n}} = ||x||^{2\left(\frac{r-1}{r}-\frac{(n-k)}{n}\right)}$$

$$\int_\Omega |f(\lambda)|^{\frac{kp}{n}}|g(\lambda)|^{\frac{(n-k)q}{n}}d\mu(\lambda) = \int_{\sigma(A)} \lambda^{\frac{k}{n}r} d\langle E(\lambda)x, x\rangle = \langle A^{\frac{k}{n}r}x, x\rangle.$$

Then by Lemma 2.2

$$\begin{aligned}\langle Ax, x\rangle &\leq \Big(\Big(\frac{1}{r}\Big)^n + \Big(\frac{r-1}{r}\Big)^n\Big)\langle A^r x, x\rangle^{\frac{1}{r}}||x||^{2(\frac{r-1}{r})} \\ &+ \sum_{k=1}^{n-1}\binom{k}{n}\Big(\frac{1}{r}\Big)^k\Big(\frac{r-1}{r}\Big)^{(n-k)}\langle A^r x, x\rangle^{\frac{1}{r}-\frac{k}{n}}||x||^{2\left(\frac{r-1}{r}-\frac{n-k}{n}\right)}\langle A^{\frac{k}{n}r}x, x\rangle.\end{aligned}$$

Then by convexity of function $f(t) = t^r$, $r \geq 1$, we have

$$\begin{aligned}\langle Ax, x\rangle^r &\leq \Big[\Big(\Big(\frac{1}{r}\Big)^n + \Big(\frac{r-1}{r}\Big)^n\Big)\langle A^r x, x\rangle^{\frac{1}{r}}||x||^{2(\frac{r-1}{r})} \\ &+ \sum_{k=1}^{n-1}\binom{k}{n}\Big(\frac{1}{r}\Big)^k\Big(\frac{r-1}{r}\Big)^{(n-k)}\langle A^r x, x\rangle^{\frac{1}{r}-\frac{k}{n}}||x||^{2\left(\frac{r-1}{r}-\frac{n-k}{n}\right)}\langle A^{\frac{k}{n}r}x, x\rangle\Big]^r \\ &\leq \Big(\Big(\frac{1}{r}\Big)^n + \Big(\frac{r-1}{r}\Big)^n\Big)\langle A^r x, x\rangle||x||^{2(r-1)} \\ &+ \sum_{k=1}^{n-1}\binom{k}{n}\Big(\frac{1}{r}\Big)^k\Big(\frac{r-1}{r}\Big)^{(n-k)}\langle A^r x, x\rangle^{1-\frac{k}{n}r}||x||^{2\left((r-1)-\frac{n-k}{n}r\right)}\langle A^{\frac{k}{n}r}x, x\rangle^r,\end{aligned}$$

which is equivalent to

$$\begin{aligned}\langle Ax, x\rangle^r||x||^{2(1-r)} &\leq \Big(\Big(\frac{1}{r}\Big)^n + \Big(\frac{r-1}{r}\Big)^n\Big)\langle A^r x, x\rangle \\ &+ \sum_{k=1}^{n-1}\binom{k}{n}\Big(\frac{1}{r}\Big)^k\Big(\frac{r-1}{r}\Big)^{(n-k)}\langle A^r x, x\rangle^{1-\frac{k}{n}r}||x||^{-2\left(\frac{n-k}{n}r\right)}\langle A^{\frac{k}{n}r}x, x\rangle^r.\end{aligned}$$

By using the inequality (1.2), we have

$$\langle B^{r'}x, x\rangle \geq \langle Bx, x\rangle^{r'}||x||^{2(1-r')} \qquad r' \geq 1$$

if we take $B = A^{\frac{k}{n}r}$ and $r' = \frac{n}{k}$, by using the inequality (2), for all $1 \leq k \leq n-1$, we have

$$\langle A^{\frac{k}{n}r}x, x\rangle \leq \langle A^r x, x\rangle^{\frac{k}{n}}||x||^{2(1-\frac{k}{n})} \qquad r' = \tfrac{n}{k} \geq 1.$$

Therefore, we have

$$\langle A^{\frac{k}{n}r}x, x\rangle^r \leq \langle A^r x, x\rangle^{\frac{k}{n}r}||x||^{2(\frac{n-k}{n})r} \qquad r' = \tfrac{n}{k} \geq 1.$$

Then

$$\begin{aligned}\langle Ax, x\rangle^r||x||^{2(1-r)} &\leq \Big(\Big(\frac{1}{r}\Big)^n + \Big(\frac{r-1}{r}\Big)^n\Big)\langle A^r x, x\rangle \\ &+ \sum_{k=1}^{n-1}\binom{k}{n}\Big(\frac{1}{r}\Big)^k\Big(\frac{r-1}{r}\Big)^{(n-k)}\langle A^r x, x\rangle^{1-\frac{k}{n}r}||x||^{-2\left(\frac{n-k}{n}r\right)}\langle A^{\frac{k}{n}r}x, x\rangle^r \\ &\leq \langle A^r x, x\rangle.\end{aligned}$$

This ends the proof. □

The third result of this note aims to provide some improvements to inequality (1.3) due to H. Albadawi and K. Shebrawi [2].

Theorem 3.3. *Let A, B be operators in $\mathcal{B}(\mathcal{H})$, and let α be any positive real number such that $0 < \alpha < 1$. Then for $r \geq 2$ and $m \geq 1$.*

$$\begin{aligned} ||A+B||^r &\leq 2^{r-1}\Big(\alpha||\,|A|^r+|B|^r\,||^{1/m}+(1-\alpha)||\,|A^*|^r+|B^*|^r\,||^{1/m}\Big)^m \\ &\leq 2^{r-1}\Big(\alpha|||A|^r+|B|^r||+(1-\alpha)|||A^*|^r+|B^*|^r||\Big). \end{aligned}$$

Proof. We have

$$\begin{aligned} |\langle A+Bx,y\rangle|^r &\leq 2^{r-1}\Big(|\langle Ax,y\rangle|^r+|\langle Bx,y\rangle|^r\Big) \\ &\quad (by\ \ Lemma\ \ 2.6) \\ &\leq 2^{r-1}\Big(\langle|A|^{2\alpha}x,x\rangle^{\frac{r}{2}}\langle|A^*|^{2(1-\alpha)}y,y\rangle^{\frac{r}{2}}+\langle|B|^{2\alpha}x,x\rangle^{\frac{r}{2}}\langle|B^*|^{2(1-\alpha)}y,y\rangle^{\frac{r}{2}}\Big) \\ &\quad (by\ \ Lemma\ \ 2.5) \\ &\leq 2^{r-1}\Big(\langle|A|^r x,x\rangle^{\alpha}\langle|A^*|^r y,y\rangle^{1-\alpha}+\langle|B|^r x,x\rangle^{\alpha}\langle|B^*|^r y,y\rangle^{1-\alpha}\Big) \\ &\quad (by\ \ Lemma\ \ 2.3) \\ &\leq 2^{r-1}\Big(\sum_{k=0}^{m}\binom{k}{m}\alpha^k(1-\alpha)^{m-k}\Big(\langle|A|^r x,x\rangle^{\frac{k}{m}}\langle|A^*|^r y,y\rangle^{\frac{m-k}{m}} \\ &+\ \langle|B|^r x,x\rangle^{\frac{k}{m}}\langle|B^*|^r y,y\rangle^{\frac{m-k}{m}}\Big)\Big) \\ &\quad (by\ \ Lemma\ \ 2.1) \end{aligned}$$

Then

$$\begin{aligned} |\langle A+Bx,y\rangle|^r &\leq 2^{r-1}\Big(\sum_{k=0}^{m}\binom{k}{m}\alpha^k(1-\alpha)^{m-k}\Big(\langle|A|^r x,x\rangle+\langle|B|^r x,x\rangle\Big)^{\frac{k}{m}} \\ &\quad .\ \ \Big(\langle|A^*|^r y,y\rangle+\langle|B^*|^r y,y\rangle\Big)^{\frac{m-k}{m}}\Big) \\ &\quad (by\ \ Lemma\ \ 2.7) \\ &= 2^{r-1}\Big(\sum_{k=0}^{m}\binom{k}{m}\alpha^k(1-\alpha)^{m-k}\Big(\langle|A|^r+|B|^r x,x\rangle\Big)^{\frac{k}{m}} \\ &\quad .\ \ \Big(\langle|A^*|^r+|B^*|^r y,y\rangle\Big)^{\frac{m-k}{m}}\Big) \\ &= 2^{r-1}\Big(\alpha\langle|A|^r+|B|^r x,x\rangle^{1/m}(1-\alpha)\langle|A^*|^r+|B^*|^r y,y\rangle^{1/m}\Big)^m \\ &\leq 2^{r-1}\Big(\alpha\langle|A|^r+|B|^r x,x\rangle+(1-\alpha)\langle|A^*|^r+|B^*|^r y,y\rangle\Big) \\ &\quad (by\ \ convexity\ of\,function\ f(t)=t^r\ r\geq 1). \end{aligned}$$

Taking the supremum over $x, y \in \mathcal{H}$, $||x|| = ||y|| = 1$ and noticing that operators $|A^*|^r + |B^*|^r$ and $|A|^r + |B|^r$ are self-adjoint, we deduce the inequality:

$$\begin{aligned} ||A+B||^r &\leq 2^{r-1}\Big(\alpha||\,|A+B|^r\,||^{1/m}+(1-\alpha)||\,|A^*|^r+B^*|^r\,||^{1/m}\Big)^m \\ &\leq 2^{r-1}\Big(\alpha|||A|^r+|B|^r||+(1-\alpha)|||A^*|^r+|B^*|^r||\Big). \end{aligned}$$

This completes the proof. □

To enunciate our results, we introduce the set $\mathcal{F}_G$ of continuous functions $f : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ satifying the following inequality:

$$f(x^r y^{1-r}) \leq f(x)^r f(y)^{1-r} \; r \in (0,1) \;\; \forall x, y \in \mathbb{R}_+. \tag{3.1}$$

The following functions are elements of the set $\mathcal{F}_G$

(i) $f(x) = x^\beta, for \;\; all \;\; x \in \mathbb{R}_+, \;\; with \;\; \beta > 0,$

(ii) $f(x) = exp(\beta x), for \;\; all \;\; x \in \mathbb{R}_+, \;\; with \;\; \beta > 0.$

The fourth main result of this note aims to provide some improvements to the inequality (1.4) and (1.5) due to S. S. Dragomir [3].

Theorem 3.4. *Let $f \in \mathcal{F}_G$ a non-decreasing convex fonction Then for any $A, B \in \mathcal{B}(\mathcal{H})$, any $\alpha \in (0,1)$, and $n \geq 1$ we have the vector inequality:*

$$\begin{aligned} f\Big(|\langle Ax, By\rangle|^2\Big) &\leq \Big(\alpha \langle f((A^*A)^{\frac{1}{\alpha}})x, x\rangle^{1/n} + (1-\alpha)\langle f((B^*B)^{\frac{1}{1-\alpha}})y, y\rangle^{1/n}\Big)^n \\ &\leq \alpha \langle f((A^*A)^{\frac{1}{\alpha}})x, x\rangle + (1-\alpha)\langle f((B^*B)^{\frac{1}{1-\alpha}})y, y\rangle, \end{aligned}$$

for any $x, y \in \mathcal{H}$ with $||x|| = ||y|| = 1$.

In particular, we have the norm inequality

$$\begin{aligned} f(||B^*A||^2) &\leq \Big(\alpha ||f\Big((A^*A)^{\frac{1}{\alpha}}\Big)||^{1/n} + (1-\alpha)||f\Big((B^*B)^{\frac{1}{1-\alpha}}\Big)||^{1/n}\Big)^n \\ &\leq \alpha ||f\Big((A^*A)^{\frac{1}{\alpha}}\Big)|| + (1-\alpha)||f\Big((B^*B)^{\frac{1}{1-\alpha}}\Big)||. \end{aligned}$$

Proof. We have

$$\begin{aligned} |\langle B^*Ax, y\rangle|^2 &\leq ||Ax||^2 ||Bx||^2 = \langle A^*Ax, x\rangle \langle B^*By, y\rangle \\ &= \langle (A^*A)^{\frac{\alpha}{\alpha}} x, x\rangle \langle (B^*B)^{\frac{1-\alpha}{1-\alpha}} y, y\rangle \\ &\leq \langle (A^*A)^{\frac{1}{\alpha}} x, x\rangle^\alpha \langle (B^*B)^{\frac{1}{1-\alpha}} y, y\rangle^{1-\alpha} \\ &\quad (by \;\; Lemma \;\; 2.3). \end{aligned}$$

So,

$$\begin{aligned} f(|\langle B^*Ax, y\rangle|^2) &\leq f\Big(\langle (A^*A)^{\frac{1}{\alpha}} x, x\rangle^\alpha \langle (B^*B)^{\frac{1}{1-\alpha}} y, y\rangle^{1-\alpha}\Big) \\ &\leq f\Big(\langle (A^*A)^{\frac{1}{\alpha}} x, x\rangle\Big)^\alpha f\Big(\langle (B^*B)^{\frac{1}{1-\alpha}} y, y\rangle\Big)^{1-\alpha} \\ &\quad (by \; inequality \; (3.1) \; satisfied \; by \; f) \\ &\leq (\langle f\Big((A^*A)^{\frac{1}{\alpha}}\Big)x, x\rangle)^\alpha (\langle f\Big((B^*B)^{\frac{1}{1-\alpha}}\Big)y, y\rangle)^{1-\alpha} \\ &\quad (by \; Lemma \; 2.4) \\ &\leq \Big(\alpha \langle f((A^*A)^{\frac{1}{\alpha}})x, x\rangle^{1/n} + (1-\alpha)\langle f((B^*B)^{\frac{1}{1-\alpha}})y, y\rangle^{1/n}\Big)^n \\ &\leq \alpha \langle f((A^*A)^{\frac{1}{\alpha}})x, x\rangle + (1-\alpha)\langle f((B^*B)^{\frac{1}{1-\alpha}})y, y\rangle. \end{aligned}$$

Taking the supremum over $x, y \in \mathcal{H}$, $||x|| = ||y|| = 1$, and noticing that operators $f((A^*A)^{\frac{1}{\alpha}})$ and $f((B^*B)^{\frac{1}{1-\alpha}})$, are self-adjoint, we deduce the inequality. □

Acknowledgements. The authors would like to express their deep thanks to the anonymous referees for their comments and suggestions on the initial version of the manuscript which lead to the improvement of this paper.

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