Turkish J. Ineq., 4 (1) (2020), Pages 14 –24 .

Turkish Journal of
INEQUALITIES

Available online at www.tjinequality.com

BOUNDEDNESS OF FRACTIONAL INTEGRAL OPERATORS CONTAINING MITTAG-LEFFLER FUNCTIONS

LIAN CHEN[1], GHULAM FARID[2], SAAD IHSAN BUTT[3], AND SAIRA BANO AKBAR[4]

Abstract. This paper studies the fractional integral operators which contain Mittag-Leffler functions in their kernels, for s-convex functions. The bounds of sum of left and right sided definitions of these operators are obtained for s-convex functions and differentiable functions whose derivatives in absolute value are s-convex. It is proved that these operators are bounded and continuous. Furthermore bounds of these operators are presented in a Hadamard like inequality.

1. Introduction

Convex functions are useful in almost all fields of mathematics. Especially in mathematical analysis, functional analysis and optimization problems, their applications are remarkable.

Definition 1.1. A function $f : I \to \mathbb{R}$ is said to be convex function, if the following inequality holds:

$$f(ta + (1-t)b) \leq tf(a) + (1-t)f(b),$$

for all $a, b \in I$ and $t \in [0, 1]$.

A generalization of convex function defined on right half of real line is called s-convex function given as follows:

Definition 1.2. [8] Let $s \in [0, 1]$. A function $f : [0, \infty) \to \mathbb{R}$ is said to be s-convex function in the second sense if

$$f(ta + (1-t)b) \leq t^s f(a) + (1-t)^s f(b),$$

holds for all $a, b \in [0, \infty)$ and $t \in [0, 1]$.

For some recent citations and utilization of s-convex functions one can see [2, 3, 9, 19] and references therein.
Convex functions and related definitions have been widely used to develop the theory of

Key words and phrases. Convex function, s-convex function, Mittag-Leffer function, Generalized fractional integral operators.

2010 *Mathematics Subject Classification.* Primary: 26A51. Secondary: 26A33, 33E12.

inequalities and their applications. A huge amount of work by many authors had/has been dedicated to theory and applications of mathematical inequalities. An equivalent geometric interpretation of convex functions is the classical Hadamard inequality.

Theorem 1.1. *Let $f : I \to \mathbb{R}$ be a convex function defined on an interval I. Then for any $a, b \in I$ with $a < b$ the following inequality holds:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

This inequality provides the upper and lower bounds of the integral mean of a convex function. It has been studied extensively by various authors and its different versions exist in diverse fields of mathematics. Since last two decades it is under consideration from fractional calculus point of view. Sarikaya et al. gave its fractional version by using Riemann-Liouville fractional integral operators (1.7) and (1.8).

Theorem 1.2. [20] *Let $f : [a, b] \to \mathbb{R}$ be a function with $0 \leq a < b$ and $f \in L_1[a, b]$. If f is a convex function on $[a, b]$, then the following inequality for Riemann-Liouville fractional integrals holds:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\beta+1)}{2(b-a)^\beta}\left[I_{a+}^\beta f(b) + I_{b-}^\beta f(a)\right] \leq \frac{f(a)+f(b)}{2}. \tag{1.2}$$

This inequality provides upper and lower bounds of the sum of left and right sided definitions of Riemann-Liouville fractional integrals at points b and a respectively. Motivated by this fractional version of the Hadamard inequality a lot of related inequalities have been published for different known integral operators, see [7, 10, 12, 13, 15, 17] and references therein. Farid studied some bounds of Riemann-Liouville fractional integral operators in variable form, see [5].

The goal of this paper is to derive bounds of integral operators which contain Mittag-Leffler functions in their kernels in variable form. The Mittag-Leffler function is defined as follows [11]:

$$E_\alpha(t) = \sum_{n=0}^{\infty} \frac{t^n}{\Gamma(\alpha n + 1)},$$

where $t, \alpha \in \mathbb{C}$, $\Re(\alpha) > 0$ and $\Gamma(.)$ is the gamma function.

The Mittag-Leffler function is a direct generalization of the exponential function to which it reduces for $\alpha = 1$. In the solution of fractional integral equations or fractional differential equations the Mittag-Leffler function arises naturally. Due to its importance it is generalized by various authors, for some generalizations see Remark 1.1. Andrić et al. introduced the following special Mittag-Leffler function [1]:

Definition 1.3. Let $\mu, \alpha, l, \gamma, c \in \mathbb{C}$, $\Re(\mu), \Re(\alpha), \Re(l) > 0$, $\Re(c) > \Re(\gamma) > 0$ with $p \geq 0$, $\delta > 0$ and $0 < k \leq \delta + \Re(\mu)$. Then the extended generalized Mittag-Leffler function is defined by:

$$E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(t;p) = \sum_{n=0}^{\infty} \frac{\beta_p(\gamma+nk, c-\gamma)}{\beta(\gamma, c-\gamma)} \frac{(c)_{nk}}{\Gamma(\mu n + \alpha)} \frac{t^n}{(l)_{n\delta}}, \tag{1.3}$$

where β_p is defined by $\beta_p(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}e^{-\frac{p}{t(1-t)}}dt$ and $(c)_{nk} = \frac{\Gamma(c+nk)}{\Gamma(c)}$.

Derivative of the generalized Mittag-Leffler function is given in following lemma.

Lemma 1.1. [1] *If* $m \in \mathbb{N}, \omega, \mu, \alpha, l, \gamma, c \in \mathbb{C}, \Re(\mu), \Re(\alpha), \Re(l) > 0, \Re(c) > \Re(\gamma) > 0$ *with* $p \geq 0, \delta > 0$ *and* $0 < k < \delta + \Re(\mu)$*, then*

$$\left(\frac{d}{dt}\right)^m [t^{\alpha-1} E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega t^\mu; p)] = t^{\alpha-m-1} E^{\gamma,\delta,k,c}_{\mu,\alpha-m,l}(\omega t^\mu; p) \quad \Re(\alpha) > m. \tag{1.4}$$

Remark 1.1. The extended Mittag-Leffler function (1.3) produces the related functions defined in [14, 16, 18, 21, 22], (see [23]).

Fractional integral operators are very useful in the advancement of mathematical inequalities. Recently several authors have established fractional integral inequalities by utilizing different fractional integral operators [5, 7, 10, 23]. Next we give the definition of integral operator containing an extended generalized Mittag-Leffler function (1.3).

Definition 1.4. [1] Let $\omega, \mu, \sigma, l, \gamma, c \in \mathbb{C}$, $\Re(\mu), \Re(\alpha), \Re(l) > 0$, $\Re(c) > \Re(\gamma) > 0$ with $p \geq 0$, $\delta > 0$ and $0 < k \leq \delta + \Re(\mu)$. Let $f \in L_1[a,b]$ and $x \in [a,b]$. Then the generalized fractional integral operators containing Mittag-Leffler function are defined by:

$$\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a^+} f\right)(x;p) = \int_a^x (x-t)^{\alpha-1} E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(x-t)^\mu; p) f(t) dt, \tag{1.5}$$

and

$$\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,b^-} f\right)(x;p) = \int_x^b (t-x)^{\alpha-1} E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(t-x)^\mu; p) f(t) dt. \tag{1.6}$$

Remark 1.2. The operators (1.5) and (1.6) produce in particular several kinds of known fractional integral operators (see [23]).

Riemann-Liouville fractional integral operators are given as follows:

Definition 1.5. Let $f \in L_1[a,b]$. Then Riemann-Liouville fractional integral operators of order $\alpha > 0$ are defined as follows:

$$I^\alpha_{a^+} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt, \quad x > a \tag{1.7}$$

and

$$I^\alpha_{b^-} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t) dt, \quad x < b. \tag{1.8}$$

It can be noted that $\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,0,a^+} f\right)(x;0) = I^\alpha_{a^+} f(x)$ $\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,0,b^-} f\right)(x;0) = I^\alpha_{b^-} f$. From extended generalized fractional integral operators, we have

$$D_{\alpha,a^+}(x;p) := \left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a^+} 1\right)(x;p) = (x-a)^\alpha E^{\gamma,\delta,k,c}_{\mu,\alpha+1,l}(w(x-a)^\mu; p) \tag{1.9}$$

$$D_{\beta,b^-}(x;p) := \left(\epsilon^{\gamma,\delta,k,c}_{\mu,\beta,l,\omega,b^-} 1\right)(x;p) = (b-x)^\beta E^{\gamma,\delta,k,c}_{\mu,\beta+1,l}(w(b-x)^\mu; p), \tag{1.10}$$

see [6]. The aim of this paper is to produce generalized fractional integral inequalities by using the generalized fractional integral operators (1.5), (1.6) and s-convex functions. For some recent citations and utilization of s-convex functions one can see [2, 3, 9, 19] and references therein.

In the next section bounds of fractional integral operators are established. The boundedness of these operators are proved. A new Hadamard inequality for s-convex functions via generalized fractional integral operators is obtained. The results of this paper are valid for convex functions defined on right half of real line.

2. Bounds of fractional integral operators

The first result is stated and proved as follows:

Theorem 2.1. *Let $f : [a, b] \longrightarrow \mathbb{R}$ be a real valued function. If f is positive and s-convex, then for $\alpha, \beta \geq 1$, the following fractional integral inequality for generalized integral operators holds:*

$$\begin{aligned}&\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a^+}f\right)(x;p) + \left(\epsilon^{\gamma,\delta,k,c}_{\mu,\beta,l,\omega,b^-}f\right)(x;p) \\ &\leq \left(\frac{f(a)+f(x)}{s+1}\right)(x-a)D_{\alpha-1,a^+}(x;p) \\ &+ \left(\frac{f(b)+f(x)}{s+1}\right)(b-x)D_{\beta-1,b^-}(x;p), x \in [a,b].\end{aligned} \tag{2.1}$$

Proof. Let $x \in [a, b]$. Then first we observe the function f on the interval $[a, x]$, for $t \in [a, x]$ and $\alpha \geq 1$, one can has the following inequality:

$$(x-t)^{\alpha-1}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(x-t)^{\mu};p) \leq (x-a)^{\alpha-1}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(x-a)^{\mu};p). \tag{2.2}$$

As f is s-convex so for $t \in [a, x]$, we have

$$f(t) \leq \left(\frac{x-t}{x-a}\right)^s f(a) + \left(\frac{t-a}{x-a}\right)^s f(x). \tag{2.3}$$

First multiplying (2.2) and (2.3). Then integrating over $[a, x]$, we get

$$\begin{aligned}&\int_a^x (x-t)^{\alpha-1}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(x-t)^{\mu};p)f(t)dt \\ &\leq \frac{(x-a)^{\alpha-1}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(x-a)^{\mu};p)}{(x-a)^s}\{f(a)\int_a^x (x-t)^s dt + f(x)\int_a^x (t-a)^s dt\},\end{aligned}$$

and then we have

$$\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a^+}f\right)(x;p) \leq (x-a)D_{\alpha-1,a^+}(x;p)\left(\frac{f(a)+f(x)}{s+1}\right). \tag{2.4}$$

Now on the other hand we observe the function f on the interval $[x, b]$. For $t \in [x, b]$ and $\beta \geq 1$, one can has the following inequality

$$(t-x)^{\beta-1}E^{\gamma,\delta,k,c}_{\mu,\beta,l}(\omega(t-x)^{\mu};p) \leq (b-x)^{\beta-1}E^{\gamma,\delta,k,c}_{\mu,\beta,l}(\omega(b-x)^{\mu};p). \tag{2.5}$$

Again for $t \in [x, b]$ using convexity of f we have

$$f(t) \leq \left(\frac{t-x}{b-x}\right)^s f(b) + \left(\frac{b-t}{b-x}\right)^s f(x). \tag{2.6}$$

Multiplying (2.5) and (2.6), then integrating over $[x,b]$, we get

$$\int_x^b (t-x)^{\beta-1} E^{\gamma,\delta,k,c}_{\mu,\beta,l}(\omega(t-x)^{\mu};p)f(t)dt$$
$$\leq \frac{(b-x)^{\beta-1} E^{\gamma,\delta,k,c}_{\mu,\beta,l}(\omega(b-x)^{\mu};p)}{(b-x)^s}\{f(b)\int_a^x (t-x)^s dt + f(x)\int_a^x (b-t)^s dt\},$$

and then we have

$$\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\beta,l,\omega,b^-} f\right)(x;p) \leq (b-x)D_{\beta-1,b^-}(x;p)\left(\frac{f(b)+f(x)}{s+1}\right). \tag{2.7}$$

Adding (2.4) and (2.7), the required inequality (2.1) is obtained. □

Corollary 2.1. *If we set $\alpha=\beta$ in (2.1), then we get following inequality:*

$$\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a^+} f\right)(x;p) + \left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,b^-} f\right)(x;p) \tag{2.8}$$
$$\leq \left(\frac{f(a)+f(x)}{s+1}\right)(x-a)D_{\alpha-1,a^+}(x;p)$$
$$+ \left(\frac{f(b)+f(x)}{s+1}\right)(b-x)D_{\alpha-1,b^-}(x;p), x\in[a,b].$$

Corollary 2.2. *Along with assumption of Theorem 1, if $f\in L_\infty[a,b]$, then we get following inequality:*

$$\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a^+} f\right)(x;p) + \left(\epsilon^{\gamma,\delta,k,c}_{\mu,\beta,l,\omega,b^-} f\right)(x;p) \tag{2.9}$$
$$\leq \frac{||f||_\infty}{s+1}\left[(x-a)D_{\alpha-1,a^+}(x;p) + (b-x)D_{\beta-1,b^-}(x;p)\right].$$

Corollary 2.3. *If $\alpha=\beta$ in (2.9), then we get following result:*

$$\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a^+} f\right)(x;p) + \left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,b^-} f\right)(x;p) \tag{2.10}$$
$$\leq \frac{||f||_\infty}{s+1}\left[(x-a)D_{\alpha-1,a^+}(x;p) + (b-x)D_{\alpha-1,b^-}(x;p)\right].$$

Corollary 2.4. *If $s=1$ in (2.9), then we get following result:*

$$\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a^+} f\right)(x;p) + \left(\epsilon^{\gamma,\delta,k,c}_{\mu,\beta,l,\omega,b^-} f\right)(x;p) \tag{2.11}$$
$$\leq \frac{||f||_\infty}{2}\left[(x-a)D_{\alpha-1,a^+}(x;p) + (b-x)D_{\beta-1,b^-}(x;p)\right].$$

Theorem 2.2. *With the assumption of Theorem 1 if $f\in L_\infty[a,b]$, then operators defined in (1.5) and (1.6) are bounded and continuous.*

Proof. If $f\in L_\infty[a,b]$, then from (2.4) we have

$$\left|\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a^+} f\right)(x;p)\right| \leq 2||f||_\infty|x-a|D_{\alpha-1,a^+}(x;p) \leq \frac{2(b-a)D_{\alpha-1,a^+}(b;p)||f||_\infty}{s+1}. \tag{2.12}$$

That is

$$\left|\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a^+}f\right)(x;p)\right| \leq M||f||_\infty,$$

where $M = \frac{2(b-a)D_{\alpha-1,a^+}(b;p)}{s+1}$.

Therefore $\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a^+}f\right)(x;p)$ is bounded also it is easy to see that it is linear, hence this is continuous operator. Also on the other hand from (2.7) one can obtain

$$\left|\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\beta,l,\omega,b^-}f\right)(x;p)\right| \leq K||f||_\infty,$$

where $K = \frac{2(b-a)D_{\beta-1,b^-}(a;p)}{s+1}$. Therefore $\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\beta,l,\omega,b^-}f\right)(x;p)$ is bounded and linear, hence it is continuous operator. □

Next result is for functions whose derivative in absolute value are s-convex.

Theorem 2.3. *Let $f : [a,b] \longrightarrow \mathbb{R}$ be a real valued function. If f is differentiable and $|f'|$ is s-convex, then for $\alpha, \beta \geq 1$, the following fractional integral inequality for generalized integral operators (1.5) and (1.6) holds:*

$$\begin{aligned}&\left|\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha+1,l,\omega,a^+}f\right)(x;p) + \left(\epsilon^{\gamma,\delta,k,c}_{\mu,\beta+1,l,\omega,b^-}f\right)(x;p)\right.\\&\left.-\left(D_{\alpha-1,a^+}(x;p)f(a) + D_{\beta-1,b^-}(x;p)f(b)\right)\right|\\&\leq \left(\frac{|f'(a)|+|f'(x)|}{s+1}\right)(x-a)D_{\alpha-1,a^+}(x;p)\\&+\left(\frac{|f'(b)|+|f'(x)|}{s+1}\right)(b-x)D_{\beta-1,b^-}(x;p), x \in [a,b].\end{aligned} \tag{2.12}$$

Proof. As $x \in [a,b]$ and $t \in [a,x]$, by using s-convexity of $|f'|$, we have

$$|f'(t)| \leq \left(\frac{x-t}{x-a}\right)^s |f'(a)| + \left(\frac{t-a}{x-a}\right)^s |f'(x)|. \tag{2.13}$$

From (2.13), one can has

$$f'(t) \leq \left(\frac{x-t}{x-a}\right)^s |f'(a)| + \left(\frac{t-a}{x-a}\right)^s |f'(x)|. \tag{2.14}$$

The product of (2.2) and (2.14), gives the following inequality

$$\begin{aligned}&(x-t)^{\alpha-1}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(x-t)^\mu;p)f'(t)dt\\&\leq (x-a)^{\alpha-2}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(x-a)^\mu;p)\left(|f'(a)|(x-t)+|f'(x)|(t-a)\right).\end{aligned} \tag{2.15}$$

After integrating above inequality over $[a,x]$, we get

$$\begin{aligned}&\int_a^x (x-t)^{\alpha-1}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(x-t)^\mu;p)f'(t)dt\\&\leq (x-a)^{\alpha-2}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(x-a)^\mu;p)\{|f'(a)|\int_a^x(x-t)dt + |f'(x)|\int_a^x(t-a)dt\}\\&= (x-a)^{\alpha}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(x-t)^\mu;p)\left(\frac{|f'(a)|+|f'(x)|}{s+1}\right).\end{aligned} \tag{2.16}$$

The left hand side of (2.16) is calculated as follows:

$$\int_a^x (x-t)^{\alpha-1} E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(x-t)^{\mu};p)f'(t)dt, \tag{2.17}$$

put $x-t=z$ that is $t=x-z$, also using the derivative property (1.4) of Mittag-Leffler function, we have

$$\begin{aligned}&\int_0^{x-a} z^{\alpha-1} E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega z^{\mu};p)f'(x-z)dz\\ &=(x-a)^{\alpha-1}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(x-a)^{\mu};p)f(a)-\int_0^{x-a} z^{\alpha-2}E^{\gamma,\delta,k,c}_{\mu,\alpha-1,l}(\omega z^{\mu};p)f(x-z)dz,\end{aligned}$$

now put $x-z=t$ in second term of the right hand side of the above equation and then using (1.5), we get

$$\begin{aligned}&\int_0^{x-a} z^{\alpha-1} E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega z^{\mu};p)f'(x-z)dz\\ &=(x-a)^{\alpha-1}E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(x-a)^{\mu};p)f(a)-\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha+1,l,\omega,a^+}f\right)(x;p).\end{aligned}$$

Therefore (2.16) takes the following form

$$\begin{aligned}&\left(D_{\alpha-1,a^+}(x;p)\right)f(a)-\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha+1,l,\omega,a^+}f\right)(x;p)\\ &\leq (x-a)D_{\alpha-1,a^+}(x;p)\left(\frac{|f'(a)|+|f'(x)|}{s+1}\right).\end{aligned} \tag{2.18}$$

Also from (2.13), one can has

$$f'(t)\geq -\left(\left(\frac{x-t}{x-a}\right)^s|f'(a)|+\left(\frac{t-a}{x-a}\right)^s|f'(x)|\right). \tag{2.19}$$

Following the same procedure as we did for (2.14), we also have

$$\begin{aligned}&\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha+1,l,\omega,a^+}f\right)(x;p)-D_{\alpha-1,a^+}(x;p)f(a)\\ &\leq (x-a)D_{\alpha-1,a^+}(x;p)\left(\frac{|f'(a)|+|f'(x)|}{s+1}\right).\end{aligned} \tag{2.20}$$

From (2.18) and (2.20), we get

$$\begin{aligned}&\left|\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha+1,l,\omega,a^+}f\right)(x;p)-D_{\alpha-1,a^+}(x;p)f(a)\right|\\ &\leq (x-a)D_{\alpha-1,a^+}(x;p)\left(\frac{|f'(a)|+|f'(x)|}{s+1}\right).\end{aligned} \tag{2.21}$$

Now we let $x\in[a,b]$ and $t\in[x,b]$. Then by using s-convexity of $|f'|$ we have

$$|f'(t)|\leq \left(\frac{t-x}{b-x}\right)^s|f'(b)|+\left(\frac{b-t}{b-x}\right)^s|f'(x)|. \tag{2.22}$$

On the same lines as we have done for (2.2), (2.14) and (2.19) one can get from (2.5) and (1.10), the following inequality:

$$\begin{aligned}&\left|\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\beta+1,l,\omega,b^-}f\right)(x;p)-D_{\beta-1,b^-}(x;p)f(b)\right|\\ &\leq (b-x)D_{\beta-1,b^-}(x;p)\left(\frac{|f'(b)|+|f'(x)|}{s+1}\right).\end{aligned} \tag{2.23}$$

From inequalities (2.21) and (2.23) via triangular inequality (2.12) is obtained. □

Corollary 2.5. *If we put $\alpha = \beta$ in (2.12), then we get*

$$\begin{aligned}&\Big|\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha+1,l,\omega,a^+}f\right)(x;p) + \left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha+1,l,\omega,b^-}f\right)(x;p)\\ &-\left(D_{\alpha-1,a^+}(x;p)f(a) + D_{\alpha-1,b^-}(x;p)f(b)\right)\Big|\\ &\leq \left(\frac{|f'(a)|+|f'(x)|}{s+1}\right)(x-a)D_{\alpha-1,a^+}(x;p)\\ &+\left(\frac{|f'(b)|+|f'(x)|}{s+1}\right)(b-x)D_{\alpha-1,b^-}(x;p), x\in[a,b].\end{aligned} \tag{2.24}$$

It is easy to prove the next lemma which will be helpful to produce Hadamard type estimations for the generalized fractional integral operators.

Lemma 2.1. *Let $f:[a,b]\to\mathbb{R}$ be a s-convex function. If f is symmetric about $\frac{a+b}{2}$, then the following inequality holds*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{2^{s-1}}f(x). \tag{2.25}$$

Theorem 2.4. *Let $f:[a,b]\longrightarrow\mathbb{R}$, $a>b$, be a real valued function. If f is positive, s-convex and symmetric about $\frac{a+b}{2}$, then for $\alpha,\beta>0$, the following fractional integral inequality for generalized integral operators (1.5) and (1.6) holds:*

$$\begin{aligned}&2^{s-1}f\left(\frac{a+b}{2}\right)\left[D_{\beta+1,b^-}(a;p) + D_{\alpha+1,a^+}(b;p)\right]\\ &\leq \left(\epsilon^{\gamma,\delta,k,c}_{\mu,\beta+1,l,\omega,b^-}f\right)(a;p) + \left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha+1,l,\omega,a^+}f\right)(b;p)\\ &\leq \left[D_{\beta-1,b^-}(a;p) + D_{\alpha-1,a^+}(b;p)\right](b-a)^2\left(\frac{f(a)+f(b)}{s+1}\right).\end{aligned} \tag{2.26}$$

Proof. For $x\in[a,b]$, we have

$$(x-a)^\beta E^{\gamma,\delta,k,c}_{\mu,\beta,l}(\omega(x-a)^\mu;p) \leq (b-a)^\beta E^{\gamma,\delta,k,c}_{\mu,\beta,l}(\omega(b-a)^\mu;p), \beta>0 \tag{2.27}$$

As f is s-convex so for $x\in[a,b]$, we have:

$$f(x) \leq \left(\frac{x-a}{b-a}\right)^s f(b) + \left(\frac{b-x}{b-a}\right)^s f(a). \tag{2.28}$$

Multiplying (2.27) and (2.28), then integrating over $[a,b]$, we get

$$\begin{aligned}&\int_a^b (x-a)^\beta E^{\gamma,\delta,k,c}_{\mu,\beta,l}(\omega(x-a)^\mu;p)f(x)dx\\ &\leq \frac{(b-a)^\beta E^{\gamma,\delta,k,c}_{\mu,\beta,l}(\omega(b-a)^\mu;p)}{(b-a)^s}\left[f(b)\int_a^b(x-a)^s dx + f(a)\int_a^b(b-x)^s dx\right].\end{aligned}$$

From which we have

$$\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\beta+1,l,\omega,b^-}f\right)(a;p) \leq (b-a)^{\beta+1}E^{\gamma,\delta,k,c}_{\mu,\beta,l}(\omega(b-a)^\mu;p)\left(\frac{f(a)+f(b)}{s+1}\right) \tag{2.29}$$

$$\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\beta+1,l,\omega,b^-}f\right)(a;p) \leq (b-a)^2 D_{\beta-1,b^-}(a;p)\left(\frac{f(a)+f(b)}{s+1}\right). \tag{2.30}$$

Now on the other hand for $x \in [a,b]$, we have

$$(b-x)^{\beta} E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(b-x)^{\mu};p) \leq (b-a)^{\alpha} E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(b-a)^{\mu};p), \alpha > 0 \tag{2.31}$$

Multiplying (2.28) and (2.31), then integrating over $[a,b]$, we get

$$\int_a^b (b-x)^{\alpha} E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(b-x)^{\mu};p) f(x)dx$$
$$\leq \frac{(b-a)^{\alpha} E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(b-a)^{\mu};p)}{(b-a)^s} \left[f(b) \int_a^b (x-a)^s dx + f(a) \int_a^b (b-x)^s dx \right].$$

From which we have

$$\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha+1,l,\omega,a^+} f\right)(b;p) \leq (b-a)^{\alpha+1} E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(b-a)^{\mu};p) \left(\frac{f(a)+f(b)}{s+1}\right) \tag{2.32}$$

$$\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha+1,l,\omega,a^+} f\right)(b;p) \leq (b-a)^2 D_{\alpha-1,a^+}(b;p) \left(\frac{f(a)+f(b)}{s+1}\right). \tag{2.33}$$

Adding (2.30) and (2.33), we get

$$\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\beta+1,l,\omega,b^-} f\right)(a;p) + \left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha+1,l,\omega,a^+} f\right)(b;p) \tag{2.34}$$
$$\leq \left[D_{\beta-1,b^-}(a;p) + D_{\alpha-1,a^+}(b;p)\right](b-a)^2 \left(\frac{f(a)+f(b)}{s+1}\right).$$

Multiplying (2.25) by $(x-a)^{\beta} E^{\gamma,\delta,k,c}_{\mu,\beta,l}(\omega(x-a)^{\mu};p)$ and integrating over $[a,b]$

$$f\left(\frac{a+b}{2}\right) \int_a^b (x-a)^{\beta} E^{\gamma,\delta,k,c}_{\mu,\beta,l}(\omega(x-a)^{\mu};p)dx \tag{2.35}$$
$$\leq \frac{1}{2^{s-1}} \int_a^b (x-a)^{\beta} E^{\gamma,\delta,k,c}_{\mu,\beta,l}(\omega(x-a)^{\mu};p) f(x)dx.$$

By using (1.6) and (1.10), we get

$$f\left(\frac{a+b}{2}\right) D_{\beta+1,b^-}(a;p) \leq \frac{1}{2^{s-1}} \left(\epsilon^{\gamma,\delta,k,c}_{\mu,\beta+1,l,\omega,b^-} f\right)(a;p). \tag{2.36}$$

Multiplying (2.25) with $(b-x)^{\alpha} E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(b-x)^{\mu};p)$ and integrating over $[a,b]$, also using (1.5) and (1.9), we get

$$f\left(\frac{a+b}{2}\right) D_{\alpha+1,a^+}(b;p) \leq \frac{1}{2^{s-1}} \left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha+1,l,\omega,a^+} f\right)(b;p). \tag{2.37}$$

By adding (2.36) and (2.37), we get;

$$2^{s-1} f\left(\frac{a+b}{2}\right) \left[D_{\beta+1,b^-}(a;p) + D_{\alpha+1,a^+}(b;p)\right] \tag{2.38}$$
$$\leq \left(\epsilon^{\gamma,\delta,k,c}_{\mu,\beta+1,l,\omega,b^-} f\right)(a;p) + \left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha+1,l,\omega,a^+} f\right)(b;p).$$

By adding (2.34) and (2.38), inequality (2.26) is obtained. □

Corollary 2.6. *If we put $\alpha = \beta$ in (2.26), then we get*

$$2^{s-1} f\left(\frac{a+b}{2}\right)\left[D_{\alpha+1,b^-}(a;p) + D_{\alpha+1,a^+}(b;p)\right] \tag{2.39}$$
$$\leq \left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha+1,l,\omega,b^-} f\right)(a;p) + \left(\epsilon^{\gamma,\delta,k,c}_{\mu,\alpha+1,l,\omega,a^+} f\right)(b;p)$$
$$\leq \left[D_{\alpha-1,b^-}(a;p) + D_{\alpha-1,a^+}(b;p)\right](b-a)^2\left(\frac{f(a)+f(b)}{s+1}\right).$$

Concluding Remarks

This work deals with the boundedness of generalized fractional integral operators given in (1.5) and (1.6), by using s-convex functions. The results of this paper provide the boundedness and continuity of several known integral operators defined in [14, 16, 18, 21, 22]. By applying s-convexity of functions f and $|f'|$ variable bounds of sum of left and right definitions of these operators are obtained, while by imposing an additional condition of symmetry a Hadamard inequality is proved. All the results hold for convex functions and for integral operators given in [14, 16, 18, 21, 22]. The method adopted in this paper can be applied to derive bounds of other kinds of well known integral operators already exist in literature.

Conflict of interest. Authors do not have any conflict of interest.

References

[1] M. Andrić, G. Farid and J. Pečarić, *A further extension of Mittag-Leffler function*, Fract. Calc. Appl. Anal., **21**(5) (2018), 1377–1395.

[2] M. Alomari, M. Darus, *The Hadamard's inequality for s-convex functions of 2-variables*, Int. J. Math. Anal., **2**(13) (2008), 629–638.

[3] S. S. Dragomir, S. Fitzpatrick, *The Hadamard's inequality for s-convex functions in the second sense*, Demonstr. Math., **32**(4) (1999), 687–696.

[4] G. Farid, K. A. Khan, N. Latif, A. U. Rehman and S. Mehmood, *General fractional integral inequalities for convex and m-convex functions via an extended generalized Mittag-Leffler function*, J. Inequal. Appl., **2018** (2018), 2018:243.

[5] G. Farid, *Some Riemann-Liouville fractional integral inequalities for convex functions*, J. Anal., (2018), doi.org/10.1007/s41478-0079-4.

[6] G. Farid, K. A. Khan, N. Latif, A. U. Rehman and S. Mehmood, *General fractional integral inequalities for convex and m-convex functions via an extended generalized Mittag-Leffler function*, J. Inequal. Appl., **2018** (2018), 2018:243.

[7] G. Farid, U. N. Katugampola, M. Usman, *Ostrowski type fractional integral inequalities for s-Godunova-Levin functions via Katugampola fractional integrals*, Open J. Math. Sci., **1**(1) (2017), 97–110.

[8] H. Hudzik, L. Maligranda, *Some remarks on s-convex functions*, Aequ. Math., **48** (1994), 100–111.

[9] S. Hussain, M. I. Bhatti, M. Iqbal, *Hadamard-type inequalities for s-convex functions*, Punjab Univ. J. Math., **41** (2009), 51–60.

[10] S. M. Kang, G. Farid, W. Nazeer, B. Tariq, *Hadamard and Fejer-Hadamard inequalities for extended generalized fractional integrals involving special functions*, J. Inequal. Appl., **2018** (2018), 2018:119.

[11] G. M. Mittag-Leffler, *Sur la nouvelle fonction $E_\alpha(x)$*, C. R. Acad. Sci., Paris, **137** (1903), 554–558.

[12] K. S. Nisar, A. Tassaddiq, G. Rahman, A. Khan, *Some inequalities via fractional conformable integral operators*, J. Inequal. Appl., **2019** (2019), 2019:217.

[13] K. S. Nisar, G. Rahman, K. Mehrez, *Chebyshev type inequalities via generalized fractional conformable integrals*, J. Inequal. Appl., **2019** (2019), 2019:245.

[14] T. R. Prabhakar, *A singular integral equation with a generalized Mittag-Leffler function in the kernel*, Yokohama Math. J., **19** (1971), 7–15.
[15] G. Rahman, T. Abdeljawad, F. Jarad, A. Khan, K. S. Nisar, *Certain inequalities via generalized proportional Hadamard fractional integral operators*, Adv. Diff. Equ., **2019** (2019), 454.
[16] G. Rahman, D. Baleanu, M. A. Qurashi, S. D. Purohit, S. Mubeen and M. Arshad, *The extended Mittag-Leffler function via fractional calculus*, J. Nonlinear Sci. Appl., **10** (2017), 4244–4253.
[17] G. Rahman, A. Khan, T. Abdeljawad, K. S. Nisar, *The Minkowski inequalities via generalized proportional fractional integral operators*, Adv. Diff. Equ., **2019** (2019), 287.
[18] T. O. Salim and A. W. Faraj, *A Generalization of Mittag-Leffler function and integral operator associated with integral calculus*, J. Frac. Calc. Appl., **3**(5) (2012), 1–13.
[19] M. Z. Sarikaya, E. Set, M. E. Özdemir, *On new inequalities of Simpson's type for s-convex functions*, Comput. Math. Appl., **60** (2010), 2191–2199.
[20] M. Z. Sarikaya, E. Set, H. Yaldiz, N. Basak, *Hermite-Hadamard's inequalities for fractional integrals and related fractional inequalities*, Math. Comp. Modelling, **57** (2013), 2403–2407.
[21] A. K. Shukla and J. C. Prajapati, *On a generalization of Mittag-Leffler function and its properties*, J. Math. Anal. Appl., **336** (2007), 797–811.
[22] H. M. Srivastava and Z. Tomovski, *Fractional calculus with an integral operator containing generalized Mittag-Leffler function in the kernel*, Appl. Math. Comput., **211**(1) (2009), 198–210.
[23] S. Ullah, G. Farid , K. A. Khan, A. Waheed and S. Mehmood, *Generalized fractional inequalities for quasi-convex functions*, Adv. Difference Equ., **2019** (2019), 15.

[1]Institute of Computing Science and Technology,
Guangzhou University,
Guangzhou 510006, China,
E-mail address: `chenlian@gzhu.edu.cn`

[2]Department of Mathematics,
COMSATS University Islamabad,
Attock Campus, Pakistan
E-mail address: `faridphdsms@hotmail.com`, `ghlmfarid@cuiatk.edu.pk`

[3]Department of Mathematics,
COMSATS University Islamabad,
Lahore Campus, Pakistan
E-mail address: `saadihsanbutt@cuilahore.edu.pk`

[4]Department of Mathematics,
COMSATS University Islamabad,
Lahore Campus, Pakistan
E-mail address: `sairabano644@yahoo.com`