Turkish J. Ineq., 3 (2) (2019), Pages 79 –87 .

Turkish Journal of
INEQUALITIES

Available online at www.tjinequality.com

SOME INTEGRAL INEQUALITIES FOR SYMMETRIZED p-CONVEX FUNCTIONS

TÜLIN NAL KARADEMIR[1], İMDAT İŞCAN[2], AND MAHIR KADAKAL[2]

ABSTRACT. In this paper, new inequalities and generalizations have been made especially for p-convex functions. New inequalities related to products of functions which is a more general form of the class of symmetric convex functions are obtained. The inequalities obtained have been shown to be compatible with the literature.

1. Preliminaries

Let real function f be defined on some nonempty interval I of real line $\mathbb{R}$. A function $f : I \to \mathbb{R}$ is said to be convex if the inequality

$$f\left(tx+(1-t)y\right) \leq tf\left(x\right)+(1-t)\,f\left(y\right)$$

is valid for all $x, y \in I$ and $t \in [0, 1]$. If this inequality reverses, then f is said to be concave on interval $I \neq \emptyset$.

Convexity theory provides powerful principles and techniques to study a wide class of problems in both pure and applied mathematics.

Let $f : I \to \mathbb{R}$ be a convex function. Then the following Hermite-Hadamard inequalities hold

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f\left(a\right)+f(b)}{2} \tag{1.1}$$

for all $a, b \in I$ with $a < b$. Both inequalities hold in the reversed direction if the function f is concave. This double inequality is well known as the Hermite-Hadamard inequality [5]. Some refinements of the Hermite-Hadamard inequality for convex functions have been obtained [1, 15]. Note that some of the classical inequalities for means can be derived from Hermite-Hadamard integral inequalities for appropriate particular selections of the mapping f. In recent years, many new convex classes and related Hermite Hadamard type inequalities have been studied by many authors (for example see [1, 2, 7, 9, 11–13, 15]).

Key words and phrases. Convexity, Hermite-Hadamard integral inequalities, p-convex functions, symmetrized p-convexity, harmonic convexity

2010 *Mathematics Subject Classification.* Primary:26A51. Secondary:26D10, 26D15.

Received: 13/11/2019

Accepted: 11/12/2019.

Definition 1.1 ([6]). Let $I \subset \mathbb{R}\backslash\{0\}$ be a interval. A function $f : I \to \mathbb{R}$ is said to be harmonically convex, if

$$f\left(\frac{xy}{tx+(1-t)y}\right) \leq tf(y)+(1-t)f(x)$$

for all $x, y \in I$ and $t \in [0,1]$. If the above inequality is reversed, then f is said to be harmonically concave.

In [7], the author gave the definition of p-convex function as follow:

Definition 1.2 ([7]). Let $I \subset (0,\infty)$ be a real interval and $p \in \mathbb{R}\backslash\{0\}$. A function $f : I \to \mathbb{R}$ is said to be a p-convex function, if

$$f\left([a^p+b^p-x^p]^{\frac{1}{p}}\right) \leq tf(x)+(1-t)f(y) \tag{1.2}$$

for all $x, y \in I$ and $t \in [0,1]$. If the inequality in (1.2) is reversed, then f is said to be p-concave.

For a function $f : [a,b] \to \mathbb{C}$ we consider the *symmetrical transform* of f on the interval $[a,b]$; denoted by $\widetilde{f}_{[a,b]}$ or simply $\widetilde{f}$, when the interval $[a,b]$ is implicit, which is defined by

$$\widetilde{f}(t) := \frac{1}{2}\left[f(t)+f(a+b-t)\right],\ t \in [a,b].$$

The *anti-symmetrical transform* of f on the interval $[a,b]$ is denoted by $\overline{f}_{[a,b]}$, or simply $\overline{f}$ and is defined by

$$\overline{f}(t) := \frac{1}{2}\left[f(t)-f(a+b-t)\right],\ t \in [a,b].$$

It is obvious that for any function f we have $\widetilde{f}+\overline{f}=f$ [3].

Definition 1.3 ([3]). We say that the function $f : [a,b] \to \mathbb{R}$ is symmetrized convex (concave) on the interval $[a,b]$ if the symmetrical transform $\widetilde{f}$ is convex (concave) on $[a,b]$.

Now, for a function $f : [a,b] \subset \mathbb{R}\backslash\{0\} \to \mathbb{C}$ we consider the *symmetrical transform* of f on the interval $[a,b]$; denoted by $\widetilde{f}_{H,[a,b]}$ or simply $\widetilde{f}_H$, when the interval $[a,b]$ is implicit, as defined by

$$\widetilde{f}_H(t) := \frac{1}{2}\left[f(t)+f\left(\frac{abt}{(a+b)\,t-ab}\right)\right],\ t \in [a,b].$$

The *anti-symmetrical transform* of f on the interval $[a,b]$ is denoted by $\overline{f}_{H,[a,b]}$, or simply $\overline{f}_H$ and is defined by

$$\overline{f}_H(t) := \frac{1}{2}\left[f(t)-f\left(\frac{abt}{(a+b)\,t-ab}\right)\right],\ t \in [a,b].$$

It is obvious that for any function f we have $\widetilde{f}_H+\overline{f}_H=f$ [14].

Definition 1.4 ([14]). A function $f : I \subset \mathbb{R}\backslash\{0\} \to \mathbb{R}$ is said to be symmetrized harmonic convex (concave) on I if $\widetilde{f}_H$ is harmonic convex (concave) on I.

For a function $f:[a,b]\subseteq(0,\infty)\to\mathbb{R}$ we consider the p-symmetrical transform of f on the interval, denoted by $P_{(f;p),[a,b]}$ or simply $P_{(f;p)}$, when the interval $[a,b]$ is implicit, which is defined by

$$P_{(f;p)}(x):=\frac{1}{2}\left[f(x)+f\left([a^p+b^p-x^p]^{\frac{1}{p}}\right)\right],x\in[a,b].$$

The anti p-symmetrical transform of f on the interval $[a,b]$ is denoted by $AP_{(f;p),[a,b]}$ or simply $AP_{(f;p)}$ as defined by

$$AP_{(f;p)}(x):=\frac{1}{2}\left[f(x)-f\left([a^p+b^p-x^p]^{\frac{1}{p}}\right)\right],x\in[a,b]\ \ [8].$$

Definition 1.5 ([8])**.** A function $f:[a,b]\subseteq(0,\infty)\to\mathbb{R}$ is said to be symmetrized p-convex (p-concave) on $[a,b]$ if p-symmetrical trasform $P_{(f;p)}$ is p-convex (p-concave) on $[a,b]$.

Theorem 1.1 ([8])**.** *If $f:[a,b]\subseteq(0,\infty)\to\mathbb{R}$ is symmetrized p-convex on the interval $[a,b]$, then we have the Hermite-Hadamard inequalities*

$$f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right)\leq\frac{p}{b^p-a^p}\int_a^b\frac{f(x)}{x^{1-p}}dx\leq\frac{f(a)+f(b)}{2}\tag{1.3}$$

Theorem 1.2 ([8])**.** *If $f:[a,b]\subseteq(0,\infty)\to\mathbb{R}$ is symmetrized p-convex on the interval $[a,b]$, then the following inequalities hold for all $x\in[a,b]$:*

$$f\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right)\leq P_{(f;p)}(x)\leq\frac{f(a)+f(b)}{2}\tag{1.4}$$

The following Hermite-Hadamard type inequalities for the product of two functions hold:

Theorem 1.3 ([4])**.** *Assume that both $f,g:[a,b]\to\mathbb{R}$ are symmetrized convex or symmetrized concave and integrable on the interval $[a,b]$. Then we have*

$$\begin{aligned}&\frac{1}{b-a}\int_a^b\widetilde{f}(t)\,\widetilde{g}(t)\,dt+f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)\\ \geq\ &f\left(\frac{a+b}{2}\right)\frac{1}{b-a}\int_a^b g(t)dt+g\left(\frac{a+b}{2}\right)\frac{1}{b-a}\int_a^b f(t)dt,\\ &\frac{1}{b-a}\int_a^b\widetilde{f}(t)\,\widetilde{g}(t)\,dt+\frac{f(a)+f(b)}{2}\frac{g(a)+g(b)}{2}\\ \geq\ &\frac{f(a)+f(b)}{2}\frac{1}{b-a}\int_a^b g(t)dt+\frac{g(a)+g(b)}{2}\frac{1}{b-a}\int_a^b f(t)dt\\ &\frac{f(a)+f(b)}{2}\frac{1}{b-a}\int_a^b g(t)dt+g\left(\frac{a+b}{2}\right)\frac{1}{b-a}\int_a^b f(t)dt\\ \geq\ &\frac{1}{b-a}\int_a^b\widetilde{f}(t)\,\widetilde{g}(t)\,dt+\frac{f(a)+f(b)}{2}g\left(\frac{a+b}{2}\right)\end{aligned}$$

and

$$\begin{aligned}&\frac{g(a)+g(b)}{2}\frac{1}{b-a}\int_a^b f(t)dt+f\left(\frac{a+b}{2}\right)\frac{1}{b-a}\int_a^b g(t)dt\\ \geq\ &\frac{1}{b-a}\int_a^b\widetilde{f}(t)\,\widetilde{g}(t)\,dt+\frac{g(a)+g(b)}{2}f\left(\frac{a+b}{2}\right).\end{aligned}$$

Theorem 1.4 ([4])**.** *Assume that both $f, g : [a, b] \to [0, \infty)$ are symmetrized convex (symmetrized concave) and integrable on the interval $[a, b]$. Then we have*

$$\begin{aligned} f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right) &\leq (\geq) f\left(\frac{a+b}{2}\right) \frac{1}{b-a} \int_a^b g(t)dt \\ &\leq (\geq) \frac{1}{b-a} \int_a^b \widetilde{f}(t) g(t)dt \\ &\leq (\geq) \frac{f(a)+f(b)}{2} \frac{1}{b-a} \int_a^b g(t)dt \\ &\leq (\geq) \frac{f(a)+f(b)}{2} \frac{g(a)+g(b)}{2} \end{aligned}$$

and

$$\begin{aligned} f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right) &\leq (\geq) g\left(\frac{a+b}{2}\right) \frac{1}{b-a} \int_a^b f(t)dt \\ &\leq (\geq) \frac{1}{b-a} \int_a^b \widetilde{f}(t) g(t)dt \\ &\leq (\geq) \frac{g(a)+g(b)}{2} \frac{1}{b-a} \int_a^b f(t)dt \\ &\leq (\geq) \frac{f(a)+f(b)}{2} \frac{g(a)+g(b)}{2}. \end{aligned}$$

Definition 1.6 ([10])**.** Let $p \in \mathbb{R} \backslash \{0\}$. A function $\omega : [a, b] \subseteq (0, \infty) \to \mathbb{R}$ is said to be p-symmetric with respect to $\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}$ if $\omega(x) = \omega\left(\left[a^p + b^p - x^p\right]^{\frac{1}{p}}\right)$ holds for all $x \in [a, b]$.

2. Main results

We will use the notation $M_p = \left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}$ for the sake of simplicity.

Theorem 2.1. *Assume that $f, g : [a, b] \subset [0, \infty) \to \mathbb{R}$ are two symmetrized p-convex and integrable functions, then we have*

$$\begin{aligned} &\frac{p}{b^p - a^p} \int_a^b \frac{P_{(f.p)}(x) P_{(g.p)}(x)}{x^{1-p}} dx + f(M_p) g(M_p) \\ \geq\ & f(M_p) \frac{p}{b^p - a^p} \int_a^b \frac{g(x)}{x^{1-p}} dx + g(M_p) \frac{p}{b^p - a^p} \int_a^b \frac{f(x)}{x^{1-p}} dx, \end{aligned} \tag{2.1}$$

$$\begin{aligned} &\frac{p}{b^p - a^p} \int_a^b \frac{P_{(f.p)}(x) P_{(g.p)}(x)}{x^{1-p}} dx + \frac{f(a)+f(b)}{2} \frac{g(a)+g(b)}{2} \\ \geq\ & \frac{f(a)+f(b)}{2} \frac{p}{b^p - a^p} \int_a^b \frac{g(x)}{x^{1-p}} dx + \frac{g(a)+g(b)}{2} \frac{p}{b^p - a^p} \int_a^b \frac{f(x)}{x^{1-p}} dx \end{aligned} \tag{2.2}$$

$$\begin{aligned} &\frac{f(a)+f(b)}{2} \frac{p}{b^p - a^p} \int_a^b \frac{g(x)}{x^{1-p}} dx + g(M_p) \frac{p}{b^p - a^p} \int_a^b \frac{f(x)}{x^{1-p}} dx \\ \geq\ & \frac{p}{b^p - a^p} \int_a^b \frac{P_{(f.p)}(x) P_{(g.p)}(x)}{x^{1-p}} dx + \frac{f(a)+f(b)}{2} g(M_p) \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} & \frac{g(a)+g(b)}{2}\frac{p}{b^p-a^p}\int_a^b \frac{f(x)}{x^{1-p}}dx + f(M_p)\frac{p}{b^p-a^p}\int_a^b \frac{g(x)}{x^{1-p}}dx \\ \geq\ & \frac{p}{b^p-a^p}\int_a^b \frac{P_{(f.p)}(x)P_{(g.p)}(x)}{x^{1-p}}dx + \frac{g(a)+g(b)}{2}f(M_p). \end{aligned} \tag{2.4}$$

Proof. Since $f,g:[a,b]\subset[0,\infty)\to\mathbb{R}$ are two symmetrized p-convex and integrable functions, by using (1.4)

$$\left[P_{(f.p)}(x)-f(M_p)\right]\left[P_{(g.p)}(x)-g(M_p)\right]\geq 0. \tag{2.5}$$

Thus, we obtain the following inequality:

$$P_{(f.p)}(x)P_{(g.p)}(x)+f(M_p)g(M_p)\geq P_{(f.p)}(x)g(M_p)+P_{(g.p)}(x)f(M_p). \tag{2.6}$$

Multiplying by $\frac{1}{x^{1-p}}$ to the inequality (2.6) and integrating over x on the interval $[a,b]$ and then again multiplying by $\frac{p}{b^p-a^p}$ to the obtained inequality, we get

$$\begin{aligned} & \frac{p}{b^p-a^p}\int_a^b \frac{P_{(f.p)}(x)P_{(g.p)}(x)}{x^{1-p}}dx + f(M_p)g(M_p) \\ \geq\ & g(M_p)\frac{p}{b^p-a^p}\int_a^b \frac{P_{(f.p)}(x)}{x^{1-p}}dx + f(M_p)\int_a^b \frac{P_{(g.p)}(x)}{x^{1-p}}dx \end{aligned} \tag{2.7}$$

From the definition of $P_{(f.p)}$, we write the following:

$$\begin{aligned} & \frac{p}{b^p-a^p}\int_a^b \frac{P_{(f.p)}(x)}{x^{1-p}}dx \\ =\ & \frac{1}{2}\left[\frac{p}{b^p-a^p}\int_a^b \frac{f(x)}{x^{1-p}}dx + \frac{p}{b^p-a^p}\int_a^b \frac{f\left([a^p+b^p-x^p]^{\frac{1}{p}}\right)}{x^{1-p}}dx\right]. \end{aligned} \tag{2.8}$$

By changing the variable as $u=[a^p+b^p-x^p]^{\frac{1}{p}}$ in the second integral in (2.8), we have

$$\frac{p}{b^p-a^p}\int_a^b \frac{P_{(f.p)}(x)}{x^{1-p}}dx = \frac{p}{b^p-a^p}\int_a^b \frac{f(x)}{x^{1-p}}dx. \tag{2.9}$$

and similarly,

$$\frac{p}{b^p-a^p}\int_a^b \frac{P_{(g.p)}(x)}{x^{1-p}}dx = \frac{p}{b^p-a^p}\int_a^b \frac{g(x)}{x^{1-p}}dx. \tag{2.10}$$

Substituting (2.9) and (2.10) in (2.7),

$$\begin{aligned} & \frac{p}{b^p-a^p}\int_a^b \frac{P_{(f.p)}(x)P_{(g.p)}(x)}{x^{1-p}}dx + f(M_p)g(M_p) \\ \geq\ & f(M_p)\frac{p}{b^p-a^p}\int_a^b \frac{g(x)}{x^{1-p}}dx + g(M_p)\frac{p}{b^p-a^p}\int_a^b \frac{f(x)}{x^{1-p}}dx \end{aligned}$$

is obtained. Since f and g are symmetrized p-convex functions, by using the inequality (1.4) the following inequality

$$\left(\frac{f(a)+f(b)}{2}-P_{(f.p)}(x)\right)\left(\frac{g(a)+g(b)}{2}-P_{(g.p)}(x)\right)\geq 0$$

can be written for all $x \in [a,b]$. Similar to the proof of the inequality (2.1), the inequality (2.2) is obtained. Finally, from (1.4) we can write

$$\left(\frac{f(a)+f(b)}{2} - P_{(f.p)}(x)\right)\left(P_{(g.p)}(x) - g(M_p)\right) \geq 0.$$

This inequality is equivalent the following inequality

$$\frac{f(a)+f(b)}{2}P_{(g.p)}(x) + g(M_p)P_{(f.p)}(x) \geq P_{(f.p)}(x)P_{(g.p)}(x) + \frac{f(a)+f(b)}{2}g(M_p).$$

If the roles of the functions f and g are changed then the inequalities (2.3) and (2.4) are obtained. □

Remark 2.1. If we choose $p=1$ in Theorem 2.1, then we get the results for symmetrized convex functions in Theorem 1.4. That is, the obtained results coincide with the results in [4].

Corollary 2.1. *If we choose $p=-1$ in Theorem 2.1, then we get the following results for symmetric harmonic convex:*

$$\begin{aligned}
&\frac{ab}{b-a}\int_a^b \frac{f_H(x)\,g_H(x)}{x^2}dx + f\left(\frac{2ab}{a+b}\right)g\left(\frac{2ab}{a+b}\right) \\
\geq\ & f\left(\frac{2ab}{a+b}\right)\frac{ab}{b-a}\int_a^b \frac{g(x)}{x^2}dx + g\left(\frac{2ab}{a+b}\right)\frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx, \\
&\frac{ab}{b-a}\int_a^b \frac{\widetilde{f}_H(x)\,\widetilde{g}_H(x)}{x^2}dx + \frac{f(a)+f(b)}{2}\frac{g(a)+g(b)}{2} \\
\geq\ & \frac{f(a)+f(b)}{2}\frac{ab}{b-a}\int_a^b \frac{g(x)}{x^2}dx + \frac{g(a)+g(b)}{2}\frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx, \\
&\frac{f(a)+f(b)}{2}\frac{ab}{b-a}\int_a^b \frac{g(x)}{x^2}dx + g\left(\frac{2ab}{a+b}\right)\frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx \\
\geq\ & \frac{ab}{b-a}\int_a^b \frac{P_{(f.p)}(x)\,P_{(g.p)}(x)}{x^2}dx + \frac{f(a)+f(b)}{2}g\left(\frac{2ab}{a+b}\right)
\end{aligned}$$

and

$$\begin{aligned}
&\frac{g(a)+g(b)}{2}\frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx + f\left(\frac{2ab}{a+b}\right)\frac{ab}{b-a}\int_a^b \frac{g(x)}{x^2}dx \\
\geq\ & \frac{ab}{b-a}\int_a^b \frac{P_{(f.p)}(x)\,P_{(g.p)}(x)}{x^2}dx + \frac{g(a)+g(b)}{2}f\left(\frac{2ab}{a+b}\right).
\end{aligned}$$

Theorem 2.2. *Assume that $f, g : [a, b] \subset [0, \infty) \to \mathbb{R}$ are two symmetrized p-convex and integrable functions. Then we have*

$$\begin{aligned} f(M_p)\, g(M_p) &\le f(M_p) \frac{p}{b^p - a^p} \int_a^b \frac{g(x)}{x^{1-p}} dx \\ &\le \frac{p}{b^p - a^p} \int_a^b \frac{P_{(f.p)}(x)\, g(x)}{x^{1-p}} dx \\ &\le \frac{f(a) + f(b)}{2} \frac{ab}{b-a} \int_a^b \frac{g(x)}{x^2} dx \\ &\le \frac{f(a) + f(b)}{2} \frac{g(a) + g(b)}{2} \end{aligned} \tag{2.11}$$

and

$$\begin{aligned} f(M_p)\, g(M_p) &\le g(M_p) \frac{p}{b^p - a^p} \int_a^b \frac{f(x)}{x^{1-p}} dx \\ &\le \frac{p}{b^p - a^p} \int_a^b \frac{P_{(f.p)}(x)\, g(x)}{x^{1-p}} dx \\ &\le \frac{g(a) + g(b)}{2} \frac{p}{b^p - a^p} \int_a^b \frac{f(x)}{x^2} dx \\ &\le \frac{f(a) + f(b)}{2} \frac{g(a) + g(b)}{2}. \end{aligned} \tag{2.12}$$

Proof. Since $f, g : [a, b] \subset [0, \infty) \to \mathbb{R}$ are two symmetrized p-convex and integrable functions, we write the following inequalities by using (1.4) for all $x \in [a, b]$:

$$0 \le f(M_p) \le P_{(f.p)}(x) \le \frac{f(a) + f(b)}{2} \tag{2.13}$$

and

$$0 \le g(M_p) \le P_{(g.p)}(x) \le \frac{g(a) + g(b)}{2} \tag{2.14}$$

If we multiply by $P_{(g.p)}(x)$ to the inequality (2.13), then we get

$$0 \le f(M_p)\, P_{(g.p)}(x) \le P_{(f.p)}(x)\, P_{(g.p)}(x) \le \frac{f(a) + f(b)}{2} P_{(g.p)}(x). \tag{2.15}$$

Multiplying by $\frac{1}{x^{1-p}}$ to the inequality (2.15) and integrating over x on the interval $[a, b]$ and then again multiplying by $\frac{p}{b^p - a^p}$ to the obtained inequality, we have

$$\begin{aligned} 0 &\le f(M_p) \frac{p}{b^p - a^p} \int_a^b \frac{P_{(g.p)}(x)}{x^{1-p}} dx \le \frac{p}{b^p - a^p} \int_a^b \frac{P_{(f.p)}(x)\, P_{(g.p)}(x)}{x^{1-p}} dx \\ &\le \frac{f(a) + f(b)}{2} \frac{p}{b^p - a^p} \int_a^b \frac{P_{(g.p)}(x)}{x^{1-p}} dx. \end{aligned} \tag{2.16}$$

Here, by substituting (2.10) in the inequality (2.16), we get

$$\begin{aligned} 0 &\le f(M_p) \frac{p}{b^p - a^p} \int_a^b \frac{g(x)}{x^{1-p}} dx \le \frac{p}{b^p - a^p} \int_a^b \frac{P_{(f.p)}(x)\, P_{(g.p)}(x)}{x^{1-p}} dx \\ &\le \frac{f(a) + f(b)}{2} \frac{p}{b^p - a^p} \int_a^b \frac{g(x)}{x^{1-p}} dx. \end{aligned} \tag{2.17}$$

From (1.3), we write

$$g(M_p) \leq \frac{p}{b^p - a^p} \int_a^b \frac{g(x)}{x^{1-p}} dx \leq \frac{g(a) + g(b)}{2}. \tag{2.18}$$

Multiplying by $f(M_p)$ the first inequality in (2.18) and multiplying by $\frac{f(a)+f(b)}{2}$ the second inequality in (2.18), we have

$$f(M_p) g(M_p) \leq f(M_p) \frac{p}{b^p - a^p} \int_a^b \frac{g(x)}{x^{1-p}} dx \tag{2.19}$$

and

$$\frac{f(a) + f(b)}{2} \frac{p}{b^p - a^p} \int_a^b \frac{g(x)}{x^{1-p}} dx \leq \frac{f(a) + f(b)}{2} \frac{g(a) + g(b)}{2} \tag{2.20}$$

respectively. Using (2.16), (2.19) and (2.20), the inequality (2.11) is found. Similar to the proof of the inequality (2.11), the inequality (2.12) is obtained. □

Remark 2.2. If we choose $p = 1$ in Theorem 2.2, then we get the results for symmetrical convex function in Theorem 1.3. That is, the obtained results coincide with the results in [4].

Corollary 2.2. *If we choose $p = -1$ in Theorem 2.2, then we get the following results for symmetric harmonic convex:*

$$\begin{aligned} f\left(\frac{2ab}{a+b}\right) g\left(\frac{2ab}{a+b}\right) &\leq f\left(\frac{2ab}{a+b}\right) \frac{ab}{b-a} \int_a^b \frac{g(x)}{x^2} dx \\ &\leq \frac{ab}{b-a} \int_a^b \frac{\widetilde{f}_H(x)\, g(x)}{x^2} dx \\ &\leq \frac{f(a) + f(b)}{2} \frac{ab}{b-a} \int_a^b \frac{g(x)}{x^2} dx \\ &\leq \frac{f(a) + f(b)}{2} \frac{g(a) + g(b)}{2} \end{aligned}$$

and

$$\begin{aligned} f\left(\frac{2ab}{a+b}\right) g\left(\frac{2ab}{a+b}\right) &\leq g\left(\frac{2ab}{a+b}\right) \frac{ab}{b-a} \int_a^b \frac{\widetilde{f}_H(x)}{x^2} dx \\ &\leq \frac{ab}{b-a} \int_a^b \frac{\widetilde{f}_H(x)\, g(x)}{x^2} dx \\ &\leq \frac{g(a) + g(b)}{2} \frac{ab}{b-a} \int_a^b \frac{\widetilde{f}_H(x)}{x^2} dx \\ &\leq \frac{f(a) + f(b)}{2} \frac{g(a) + g(b)}{2}. \end{aligned}$$

References

[1] S.S. Dragomir and C.E.M.Pearce,,*Selected Topics on Hermite-Hadamard Inequalities and Its Applications*, RGMIA Monograph, 2002.

[2] S.S. Dragomir, J. Pečarić and L.E. Persson, *Some inequalities of Hadamard Type*, Soochow Journal of Mathematics, **21** (3)(1005), 335–341.

[3] S.S. Dragomir,, *Symmetrized convexity and hermite-hadamard type inequalities*, Journal of Mathematical Inequalities, **10**(4) (2016), 901–918.

[4] S.S. Dragomir,, *Hermite-Hadamard type inequalities for product of symmetrized convex functions*, Rgmia Res. Rep. Coll., **20** (2017), Art. 8, 11 pp.

[5] J. Hadamard, *Étude sur les propriétés des fonctions entières en particulier d'une fonction considérée par Riemann*, J. Math. Pures Appl., **58** (1893), 171–215.

[6] İ. İşcan, *Hermite-Hadamard type inequalities for harmonically convex functions*, Hacettepe Journal of Mathematics and Statistics, **43**(6) (2014), 935–942.

[7] İ. İşcan, *Ostrowski type inequalities for p-convex functions*, New Trends in Mathematical Sciences, **4**(3) (2016), 140–150.

[8] İ. İşcan, *Symmetrized p-convexity and related some integral inequalities*, TWMS J. App. and Eng. Math., (Accepted for publication), 2019.

[9] H. Kadakal, *New Inequalities for Strongly r-Convex Functions*, Journal of Function Spaces, **2019**, Article ID 1219237, 10 pages, 2019.

[10] M. Kunt, and İ. İşcan,, *Hermite-Hadamard-Fejér type inequalities for p-convex functions via fractional integrals*, Iran J Sci Technol Trans Sci., **42** (2018),2079–2089.

[11] S. Maden, H. Kadakal, M. Kadakal and İ. İşcan, *Some new integral inequalities for n-times differentiable convex and concave functions*, Journal of Nonlinear Sciences and Applications, **10**(12) (2017), 6141–6148.

[12] S. Özcan, *Some Integral Inequalities for Harmonically (α, s)-Convex Functions*, Journal of Function Spaces, **2019** 2019), Article ID 2394021, 8 pages.

[13] S. Özcan, İ. İşcan, *Some new Hermite-Hadamard type inequalities for s-convex functions and their applications*, Journal of Inequalities and Applications, Article number: **2019**:201 (2019).

[14] S. Wu, B.R. Ali, I.A. Baloch, A.U. Haq, *Inequalities related to symmetrized harmonic convex functions*, arXiv:1711.08051v1 [math.CA] 4 Nov 2017.

[15] G. Zabandan, *A new refinement of the Hermite-Hadamard inequality for convex functions*, J. Inequal. Pure Appl. Math., **10**(2) (2009), Article ID 45.

[1]Institute of Science,
Giresun University,
28200, Giresun-Turkey
E-mail address: tulinnal@gmail.com

[2]Department of Mathematics,
Giresun University,
Faculty of Arts and Sciences, 28200, Giresun-Turkey,
E-mail address: imdati@yahoo.com
E-mail address: mahirkadakal@gmail.com