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MAJORIZATION TYPE INEQUALITIES FOR STRONGLY CONVEX FUNCTIONS

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Abstract. In this paper, we present several discrete majorization type inequalities for strongly convex functions on rectangles. Our results are the generalization and improvement of the earlier results.

1. Introduction

Convex functions has many important rule in optimization theory and have several applications in mathematics, statistics, physics and engineering. Convex functions is defined as [34]:

Definition 1.1. Suppose $\phi : \mathbf{I} \to \mathbb{R}$ is said to be convex, if

$$\phi(\zeta x_1 + (1-\zeta)y_1) \leq \zeta\phi(x_1) + (1-\zeta)\phi(y_1), \tag{1.1}$$

holds for all $x_1, y_1 \in \mathbf{I}$ and $\zeta \in [0,1]$.

Recently, many extensions, refinements, generalizations and variants for the convexity can be found in the literature. Some of them are Schur convexity [13–15] quasi-convex [18], co-ordinate convex function [19], φ-convex [21], λ-convex [22], approximately convex [25], midconvex functions [26], pseudo-convex [30], strongly convex [31], h-convex [37], delta-convex [35] and others [1–3, 16, 17, 27, 34, 38, 39, 42] etc.

Strongly convex functions can be defined as:

Definition 1.2. Let ϕ be a real valued functions and c be a positive real number. Then ϕ is said to be strongly convex with modulus c, if

$$\phi(\zeta x_1 + (1-\zeta)y_1) \leq \zeta\phi(x_1) + (1-\zeta)\phi(y_1) - c\zeta(1-\zeta)(x_1-y_1)^2, \tag{1.2}$$

holds for all $x_1, y_1 \in \mathbf{I}$ and $\zeta \in [0,1]$.

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If $\phi : \mathbf{I} \to \mathbb{R}$ is strongly convex with modulus c, then

$$\phi(\mathbf{x}) - \phi(\mathbf{y}) \geq \nabla_+\phi(\mathbf{y})(\mathbf{x}-\mathbf{y}) + c(\mathbf{x}-\mathbf{y})^2, \ \ \forall \ \mathbf{x}, \mathbf{y} \in \mathbf{I}, \tag{1.3}$$

where

$$\nabla_+\phi(\mathbf{y})(\mathbf{x}-\mathbf{y}) = \left\langle \frac{\partial\phi_+(\mathbf{y})}{\partial\mathbf{y}}, (\mathbf{x}-\mathbf{y}) \right\rangle$$

and

$$\frac{\partial\phi_+(\mathbf{y})}{\partial\mathbf{y}} = \left(\frac{\partial\phi_+(\mathbf{y})}{\partial y_1}, \frac{\partial\phi_+(\mathbf{y})}{\partial y_2}, ..., \frac{\partial\phi_+(\mathbf{y})}{\partial y_n} \right),$$

for $\mathbf{x} = (x_1, x_2, ..., x_n)$, $\mathbf{y} = (y_1, y_2, ..., y_n) \in \mathbf{I}$ and $\langle ., . \rangle$ is the ordinary inner product in $\mathbb{R}^n$.

Adil Khan et al., [6], gave the concept of co-ordinate strongly convex functions, defined as:

Definition 1.3. Let $\mathbf{I}_1 \times \mathbf{I}_2 \subset \mathbb{R}^2$; a function $\phi : \mathbf{I}_1 \times \mathbf{I}_2 \to \mathbb{R}$, be coordinate strongly convex if the partial mappings $\phi_y : \mathbf{I}_1 \to \mathbb{R}$ defined as $\phi_y(u) = \phi(u, y)$, for all y $\in \mathbf{I}_2$ and $\phi_x : \mathbf{I}_2 \to \mathbb{R}$ defined as $\phi_x(v) = \phi(x, v)$, for all x $\in \mathbf{I}_1$,are strongly convex.

Remark 1.1. Adil Khan et al., proved that every strongly convex function defined on rectangle is co-ordinate strongly convex, but the converse is not true in general [6].

In the last of this section, we give some introduction about the theory of majorization. For fixed $n \geq 2$, let $\mathbf{a} = (a_1, a_2, ..., a_n)$ and $\mathbf{b} = (b_1, b_2, ..., b_n)$ be two n-tuples such that

$$a_{[1]} \geq a_{[2]} \geq ... \geq a_{[n]}, \ \ b_{[1]} \geq b_{[2]} \geq ... \geq b_{[n]}$$

be their ordered arrangement.

Definition 1.4. The n-tuple $\mathbf{a}$ is said to majorized by the n-tuple $\mathbf{b}$ or $\mathbf{b}$ majorizes $\mathbf{a}$, in symbols $\mathbf{a} \succ \mathbf{b}$, if

$$\sum_{i=1}^{k} a_{[i]} \ \ \geq \ \ \sum_{i=1}^{k} b_{[i]} \ \ for \ k = 1, 2, ..., n-1, \tag{1.4}$$

$$\sum_{i=1}^{n} b_i \ \ = \ \ \sum_{i=1}^{n} a_i. \tag{1.5}$$

In the literature a well-know theorem of majorization and its proof we refer Marshall and Olkin [32]. One can also see the paper of Hardy, Littlewood and Pólya [24] and [28].

Theorem 1.1. *Let* $\boldsymbol{a} = (a_1, a_2, ..., a_n)$ *and* $\boldsymbol{b} = (b_1, b_2, ..., b_n)$ *be two* $n-$*tuples such that* $a_i, b_i \in \mathbf{I}$ $(i = 1, 2, ..., n)$. *Then the inequality*

$$\sum_{i=1}^{n} \phi(a_i) \geq \sum_{i=1}^{n} \phi(b_i) \tag{1.6}$$

holds for every ϕ *continuous convex function if and only if* $\boldsymbol{a} \succ \boldsymbol{b}$.

In Fuchs [23] gave a weighted version of Theorem 1.1.

Theorem 1.2. *Let $\boldsymbol{a} = (a_1, a_2, ..., a_n)$, $\boldsymbol{b} = (b_1, b_2, ..., b_n)$ be two decreasing $n-$tuples such that $a_i, b_i \in \mathbf{I}$ $(\mathbf{i} = \mathbf{1}, \mathbf{2}, ..., \mathbf{n})$, and $\boldsymbol{p} = (p_1, p_2, ..., p_n)$ be a real $n-$tuple with*

$$\sum_{i=1}^{k} p_i a_i \geq \sum_{i=1}^{k} p_i b_i, \quad k = 1, 2, ..., n-1, \tag{1.7}$$

$$\sum_{i=1}^{n} p_i a_i = \sum_{i=1}^{n} p_i b_i. \tag{1.8}$$

Then for every ϕ continuous convex function, the following inequality holds

$$\sum_{i=1}^{n} p_i \phi(a_i) \geq \sum_{i=1}^{n} p_i \phi(b_i). \tag{1.9}$$

Dragomir [20] presented majorization result, by using support line inequality and Chebyshev's inequality.

Theorem 1.3. *Let $\phi : \mathbf{I} \to \mathbb{R}$ be a convex function and $\boldsymbol{a} = (a_1, a_2, ..., a_n)$, $\boldsymbol{b} = (b_1, b_2, ..., b_n)$ be two real $n-$tuples such that $a_i, b_i \in \mathbf{I}$ $(\mathbf{i} = \mathbf{1}, \mathbf{2}, ..., \mathbf{n})$ and $\boldsymbol{p} = (p_1, p_2, ..., p_n)$ be a non-negative real $n-$tuple with $P_n = \sum_{i=1}^{n} p_i > 0$. If $\boldsymbol{a} - \boldsymbol{b}$ and $\boldsymbol{b}$ are monotonic in the same sense, then*

$$\sum_{i=1}^{n} p_i \phi(a_i) \geq \sum_{i=1}^{n} p_i \phi(b_i) \tag{1.10}$$

holds. If ϕ is strictly convex and $p_i > 0$ $(i = 1, 2, ..., n)$, then inequalities (1.10) becomes equality if and only if $a_i = b_i$ for all $i = 1, 2, ..., n$.

Recently, Zaheer Ullah et al., [40] gave the majorization theorem for strongly convex functions.

Theorem 1.4. *Let $\phi : \boldsymbol{I} \to \mathbb{R}$ be a strongly convex function with respect to modulus c. Suppose $\boldsymbol{a} = (a_1, a_2, ..., a_n)$ and $\boldsymbol{b} = (b_1, b_2, ..., b_n)$ are n-tuples, $a_i, b_i \in \boldsymbol{I}$, $i = 1, 2, ..., n$ and the n-tuple $\boldsymbol{a}$ majorizes $\boldsymbol{b}$. Then the following inequality holds*

$$\sum_{i=1}^{n} \phi(a_i) \geq \sum_{i=1}^{n} \phi(b_i) + c \sum_{i=1}^{n} (a_i - b_i)^2. \tag{1.11}$$

For more details of convex functions, co-ordinate convex functions, strongly convex functions, majorization type results and their inequalities we suggest [4–12, 33, 36, 40, 41].

In this article, the main attention on the majorization type results for co-ordinate strongly convex functions. To extend majorization inequality for majorized tuples and establish some weighted version of majorization inequalities for certain n-tuples. For secure these results, by using Abel transformation, Chebyshev's inequality, support line inequality of strongly convex functions. Furthermore, we give Favard's type inequalities by using the generalized majorization types results.

2. Main Results

To start this section, first we give majorization inequality for strongly convex functions defined on rectangles.

Theorem 2.1. *Let* $\boldsymbol{a} = (a_1, a_2, ..., a_n)$*,* $\boldsymbol{b} = (b_1, b_2, ..., b_n)$ *be two* n*-tuples such that* $a_i, b_i \in \mathbf{I_1}$ $(i = 1, 2, ..., n)$ *and* $\boldsymbol{c} = (c_1, c_2, ..., c_m)$*,* $\boldsymbol{d} = (d_1, d_2, ..., d_m)$ *be two* m*-tuples such that* $c_j, d_j \in \mathbf{I_2}$ $(j = 1, 2, ..., m)$*, where* $\mathbf{I_1}$*,* $\mathbf{I_2}$ *be any two intervals in* $\mathbb{R}$*. If* $\boldsymbol{a} \succ \boldsymbol{b}$ *and* $\boldsymbol{c} \succ \boldsymbol{d}$*, then for every strongly convex function* $\phi : \boldsymbol{I}_1 \times \boldsymbol{I}_2 \longrightarrow \mathbb{R}$*, we have the following inequality*

$$\sum_{i=1}^{n}\sum_{j=1}^{m}\phi(a_i, c_j) \geq \sum_{i=1}^{n}\sum_{j=1}^{m}\phi(b_i, d_j) + c\sum_{i=1}^{n}\sum_{j=1}^{m}\left\{(a_i - b_i)^2 + (c_j - d_j)^2\right\}. \tag{2.1}$$

Proof. Without loss of generality, assume that $\mathbf{a}, \mathbf{b}, \mathbf{c}$ and $\mathbf{d}$ are in decreasing order and $a_i \neq b_i,\ c_j \neq d_j$ for all i and j. Suppose that

$$A_k = \sum_{i=1}^{k} a_i, \quad B_k = \sum_{i=1}^{k} b_i, \quad k = 1, 2, ..., n, \tag{2.2}$$

$$C_l = \sum_{j=1}^{l} c_j, \quad D_l = \sum_{j=1}^{l} d_j, \quad l = 1, 2, ..., m \tag{2.3}$$

and

$$A_0 = B_0 = 0, \quad C_0 = D_0 = 0. \tag{2.4}$$

Then from the definition of majorization, we have

$$A_n = B_n, \quad C_m = D_m. \tag{2.5}$$

Let $t_{i,j}$ and $s_{i,j}$ be defined by

$$t_{i,j} := \nabla\phi(a_i, b_i; c_j) = \frac{\phi(a_i, c_j) - \phi(b_i, c_j) - c(a_i - b_i)^2}{a_i - b_i},$$

$$s_{i,j} := \nabla\phi(b_i; c_j, d_j) = \frac{\phi(b_i, c_j) - \phi(b_i, d_j) - c(c_j - d_j)^2}{c_j - d_j}.$$

Then clearly we see that

$$\begin{aligned}
&\phi(a_i, c_j) - \phi(b_i, d_j) - c\left\{(a_i - b_i)^2 + (c_j - d_j)^2\right\} \\
&\quad = \phi(a_i, c_j) - \phi(b_i, c_j) + \phi(b_i, c_j) - \phi(b_i, d_j) - c\left\{(a_i - b_i)^2 + (c_j - d_j)^2\right\} \\
&\quad = \phi(a_i, c_j) - \phi(b_i, c_j) + \phi(b_i, c_j) - \phi(b_i, d_j) - c(a_i - b_i)^2 - c(c_j - d_j)^2 \\
&\quad = \phi(a_i, c_j) - \phi(b_i, c_j) - c(a_i - b_i)^2 + \phi(b_i, c_j) - \phi(b_i, d_j) - c(c_j - d_j)^2 \\
&\quad = \frac{\phi(a_i, c_j) - \phi(b_i, c_j) - c(a_i - b_i)^2}{a_i - b_i}(a_i - b_i) \\
&\qquad + \frac{\phi(b_i, c_j) - \phi(b_i, d_j) - c(c_j - d_j)^2}{c_j - d_j}(c_j - d_j) \\
&\quad = t_{i,j}(A_i - A_{i-1} - B_i + B_{i-1}) \\
&\qquad + s_{i,j}(C_j - C_{j-1} - D_j + D_{j-1}).
\end{aligned}$$

Taking summations over i and j, we have

$$\sum_{i=1}^{n}\sum_{j=1}^{m}\phi(a_i, c_j) - \sum_{i=1}^{n}\sum_{j=1}^{m}\phi(b_i, d_j) - c\sum_{i=1}^{n}\sum_{j=1}^{m}\left\{(a_i - b_i)^2 + (c_j - d_j)^2\right\}$$

$$
\begin{aligned}
&= \sum_{i=1}^{n}\sum_{j=1}^{m} t_{i,j}(A_i - A_{i-1} - B_i + B_{i-1}) + \sum_{i=1}^{n}\sum_{j=1}^{m} s_{i,j}(C_j - C_{j-1} - D_j + D_{j-1}) \\
&= \sum_{j=1}^{m}\left[\sum_{i=1}^{n} t_{i,j}(A_i - B_i) - \sum_{i=1}^{n} t_{i,j}(A_{i-1} - B_{i-1})\right] \\
&\quad + \sum_{i=1}^{n}\left[\sum_{j=1}^{m} s_{i,j}(C_j - D_j) - \sum_{j=1}^{m} s_{i,j}(C_{j-1} - D_{j-1})\right] \\
&= \sum_{j=1}^{m}\left[\sum_{i=1}^{n-1} t_{i,j}(A_i - B_i) - \sum_{i=2}^{n} t_{i,j}(A_{i-1} - B_{i-1})\right] \\
&\quad + \sum_{i=1}^{n}\left[\sum_{j=1}^{m-1} s_{i,j}(C_j - D_j) - \sum_{j=2}^{m} s_{i,j}(C_{j-1} - D_{j-1})\right] \\
&= \sum_{j=1}^{m}\left[\sum_{i=1}^{n-1} t_{i,j}(A_i - B_i) - \sum_{i=1}^{n-1} t_{i+1,j}(A_i - B_i)\right] \\
&\quad + \sum_{i=1}^{n}\left[\sum_{j=1}^{m-1} s_{i,j}(C_j - D_j) - \sum_{j=1}^{m-1} s_{i,j+1}(C_j - D_j)\right]
\end{aligned}
$$

$$
= \sum_{j=1}^{m}\left[\sum_{i=1}^{n-1} (t_{i,j} - t_{i+1,j})(A_i - B_i)\right] + \sum_{i=1}^{n}\left[\sum_{j=1}^{m-1} (s_{i,j} - s_{i,j+1})(C_j - D_j)\right]. \tag{2.6}
$$

Since ϕ is a strongly convex function on $\mathbf{I}_1 \times \mathbf{I}_2$, therefore ϕ is a co-ordinate strongly convex function on $\mathbf{I}_1 \times \mathbf{I}_2$. Therefore, $t_{i,j}$ is decreasing with respect to i for each fix j and $s_{i,j}$ is decreasing with respect to j for each fix i. Thus $t_{i,j} - t_{i+1,j} \geq 0$ for all $i \in \{1, 2, .., n-1\}$ and $s_{i,j} - s_{i,j+1} \geq 0$ for all $j \in \{1, 2..., m-1\}$. From the definition of majorization we get $A_i - B_i \geq 0$ for all $i \in \{1, 2, .., n-1\}$ and $C_j - D_j \geq 0$ for all $j \in \{1, 2..., m-1\}$. Therefore, the right hand side of (2.6) is non-negative, hence we deduced

$$
\sum_{i=1}^{n}\sum_{j=1}^{m} \phi(a_i, c_j) - \sum_{i=1}^{n}\sum_{j=1}^{m} \phi(b_i, d_j) - c\sum_{i=1}^{n}\sum_{j=1}^{m} \left\{(a_i - b_i)^2 + (c_j - d_j)^2\right\} \geq 0, \tag{2.7}
$$

which is equivalent to (2.1). □

Next, we prove some general inequality for strongly convex functions defined on rectangles.

Theorem 2.2. *Let $\mathbf{I_1}$, $\mathbf{I_2}$ be any two intervals in $\mathbb{R}$, $\boldsymbol{a} = (a_1, a_2, ..., a_n)$ and $\boldsymbol{b} = (b_1, b_2, ..., b_n)$ be two n-tuples such that $a_i, b_i \in \mathbf{I_1}$ $(i = 1, 2, ..., n)$, $\boldsymbol{c} = (c_1, c_2, ..., c_m)$ and $\boldsymbol{d} = (d_1, d_2, ..., d_m)$ be two m-tuples such that $c_j, d_j \in \mathbf{I_2}$ $(j = 1, 2, ..., m)$, and $\boldsymbol{p} = (p_1, p_2, ..., p_n)$ and $\boldsymbol{w} = (w_1, w_2, ..., w_m)$ be any positive real n and m-tuples respectively. If $\phi : \mathbf{I_1} \times \mathbf{I_2} \longrightarrow \mathbb{R}$ is a* *strongly convex function, then the inequality*

$$\sum_{i=1}^{n}\sum_{j=1}^{m} p_i w_j \phi(a_i, c_j) - \sum_{i=1}^{n}\sum_{j=1}^{m} p_i w_j \phi(b_i, d_j)$$
$$\geq \sum_{i=1}^{n}\sum_{j=1}^{m} p_i w_j t_i (a_i - b_i) + \sum_{i=1}^{n}\sum_{j=1}^{m} p_i w_j s_j (c_j - d_j) + c \sum_{i=1}^{n}\sum_{j=1}^{m} p_i w_j \left\{(a_i - b_i)^2 + (c_j - d_j)^2\right\}. \tag{2.8}$$

holds, where t_i is the positive partial derivative of ϕ with respect to first variable at b_i $(i = 1, 2, ..., n)$ and s_j is the positive partial derivative of ϕ with respect to second variable at d_j $(j = 1, 2, ..., m)$.

Proof. Since $\phi : \mathbf{I_1} \times \mathbf{I_2} \longrightarrow \mathbb{R}$ is a strongly convex function, therefore we have

$$\phi(x, y) - \phi(w, z) \geq \langle \nabla\phi(w, z), (x - w, y - z)\rangle + c\left\{(x - w)^2 + (y - z)^2\right\},$$

for all $(x, y), (w, z) \in \mathbf{I_1} \times \mathbf{I_2}$, that is

$$\phi(x, y) - \phi(w, z)$$
$$\geq \frac{\partial\phi}{\partial w}(w, z)(x - w) + \frac{\partial\phi}{\partial z}(w, z)(y - z) + c\left\{(x - w)^2 + (y - z)^2\right\}. \tag{2.9}$$

Now, applying (2.9) by choosing $x \to a_i$, $y \to c_j$, $w \to b_i$ and $z \to d_j$, we get

$$\phi(a_i, c_j) - \phi(b_i, d_j) \geq t_i(a_i - b_i) + s_j(c_j - d_j) + c\left\{(a_i - b_i)^2 + (c_j - d_j)^2\right\}. \tag{2.10}$$

Multiplying both sides of (2.10) by $p_i w_j$ and taking summation twice, we obtain inequality (2.8). □

To give some majorizations type results with the help of above theorem are given in the form of the following propositions.

Proposition 2.1. *Let all the hypotheses of Theorem 2.2 hold and **b**, **a-b** and **c**, **c-d** be monotonic in the same sense, such that*

$$\sum_{i=1}^{n} a_i p_i = \sum_{i=1}^{n} b_i p_i \tag{2.11}$$

and

$$\sum_{j=1}^{m} c_j w_j = \sum_{j=1}^{m} d_j w_j. \tag{2.12}$$

Then

$$\sum_{i=1}^{n}\sum_{j=1}^{m} p_i w_j \phi(a_i, c_j) \geq \sum_{i=1}^{n}\sum_{j=1}^{m} p_i w_j \phi(b_i, d_j) + \sum_{i=1}^{n}\sum_{j=1}^{m} p_i w_j \left\{(a_i - b_i)^2 + (c_j - d_j)^2\right\}. \tag{2.13}$$

Proof. Since ϕ is a strongly convex function on $\mathbf{I}_1 \times \mathbf{I}_2$, therefore ϕ is a co-ordinate strongly convex function on $\mathbf{I}_1 \times \mathbf{I}_2$. If $\mathbf{b}$ is an increasing n-tuple, then $(t_1, t_2, ..., t_n)$ is an increasing n-tuple, where t_i is the positive partial derivative of ϕ with respect to first variable at

$b_i\,(i=1,2,...,n)$. If $\mathbf{b}$ and $\mathbf{a}$-$\mathbf{b}$ are increasing n-tuples, then applying Chebyshev's inequality to first term on right hand side of (2.8) and using (2.11), we have

$$\begin{aligned}\sum_{i=1}^{n}\sum_{j=1}^{m}p_iw_jt_i(a_i-b_i) &= \sum_{j=1}^{m}w_j\left[\sum_{i=1}^{n}p_it_i(a_i-b_i)\right]\\ &\geq \sum_{j=1}^{m}w_j\left[\frac{1}{P_n}\sum_{i=1}^{n}p_it_i\sum_{i=1}^{n}p_i(a_i-b_i)\right]=0.\end{aligned} \tag{2.14}$$

Similarly, since ϕ is a co-ordinate strongly convex function on $\mathbf{I}_1\times\mathbf{I}_2$. If $\mathbf{d}$ is an increasing m-tuple, then $(s_1,s_2,...,s_m)$ is an increasing m-tuple, where s_j is the positive partial derivative of ϕ with respect to second variable at $d_j\,(j=1,2,...,m)$. If $\mathbf{d}$ and $\mathbf{c}$-$\mathbf{d}$ are increasing m-tuples, then applying Chebyshev's inequality to second term on right hand side of (2.8) and using (2.12), we have

$$\begin{aligned}\sum_{i=1}^{n}\sum_{j=1}^{m}p_iw_js_j(c_j-d_j) &= \sum_{i=1}^{n}p_i\left[\sum_{j=1}^{m}w_js_j(c_j-d_j)\right]\\ &\geq \sum_{i=1}^{n}p_i\left[\frac{1}{W_m}\sum_{j=1}^{m}w_js_j\sum_{j=1}^{m}w_j(c_j-d_j)\right]=0.\end{aligned} \tag{2.15}$$

Using (2.14) and (2.15) in (2.8), we get inequality (2.13).

Similarly, we can prove inequality (2.13) in the remaining case. □

Remark 2.1. A strongly convex functions are said to be monotonic increasing, if it is monotonic increasing with respect to its each variable.

Proposition 2.2. *Let all the assumptions of Theorem 2.2 hold. Moreover, if* $\phi:\mathbf{I_1}\times\mathbf{I_2}\longrightarrow\mathbb{R}$ *is an increasing strongly convex function and* $\boldsymbol{b}$, $\boldsymbol{a}$-$\boldsymbol{b}$ *and* $\boldsymbol{d}$, $\boldsymbol{c}$-$\boldsymbol{d}$ *are monotonic in the same sense, such that*

$$\sum_{i=1}^{n}a_ip_i\geq\sum_{i=1}^{n}b_ip_i \tag{2.16}$$

and

$$\sum_{j=1}^{m}c_jw_j\geq\sum_{j=1}^{m}d_jw_j. \tag{2.17}$$

Then inequality (2.13) *holds.*

Proof. Since ϕ is an increasing function on $\mathbf{I}_1\times\mathbf{I}_2$, therefore $t_i\geq 0\ (i=1,2,...,n)$, where t_i is the positive partial derivative of ϕ with respect to first variable at $b_i\,(i=1,2,...,n)$, thus

$$\sum_{i=1}^{n}p_it_i\geq 0. \tag{2.18}$$

Using (2.16) and (2.18) in the right hand side of (2.14), we have

$$\sum_{i=1}^{n}\sum_{j=1}^{m}p_iw_jt_i(a_i-b_i)\geq 0. \tag{2.19}$$

Similarly, we have

$$\sum_{i=1}^{n}\sum_{j=1}^{m} p_i w_j s_j (c_j - d_j) \geq 0. \tag{2.20}$$

Using (2.19) and (2.20) in (2.8), we get (2.13).

Similarly, we can prove inequality (2.13) in the remaining cases. □

The following theorem is another weighted discrete version of majorization theorem.

Theorem 2.3. *Let $\boldsymbol{I}_1$, $\boldsymbol{I}_2$ be two intervals in $\mathbb{R}$ and $\phi : \mathbf{I_1} \times \mathbf{I_2} \longrightarrow \mathbb{R}$ be a strongly convex function. Also let $\boldsymbol{a} = (a_1, a_2, ..., a_n)$ and $\boldsymbol{b} = (b_1, b_2, ..., b_n)$ be two n-tuples such that $a_i, b_i \in \mathbf{I_1}$ $(i = 1, 2, ..., n)$, $\boldsymbol{c} = (c_1, c_2, ..., c_m)$ and $\boldsymbol{d} = (d_1, d_2, ..., d_m)$ be two m-tuples such that $c_j, d_j \in \mathbf{I_2}$ $(j = 1, 2, ..., m)$, $\boldsymbol{p} = (p_1, p_2, ..., p_n)$ and $\boldsymbol{w} = (w_1, w_2, ..., w_n)$ be two any positive real n and m-tuples respectively, such that*

$$\sum_{i=1}^{k} a_i p_i \geq \sum_{i=1}^{k} b_i p_i \qquad for \;\; k = 1, 2, ..., n-1, \tag{2.21}$$

$$\sum_{i=1}^{k} c_j w_j \geq \sum_{i=1}^{k} d_j w_j \qquad for \;\; k = 1, 2, ..., m-1 \tag{2.22}$$

and

$$\sum_{i=1}^{n} a_i p_i = \sum_{i=1}^{n} b_i p_i, \tag{2.23}$$

$$\sum_{i=1}^{m} c_j w_j = \sum_{i=1}^{m} d_j w_j. \tag{2.24}$$

Then the following statements are true:

- *(i) If $\boldsymbol{b}$ and $\boldsymbol{d}$ are decreasing n and m-tuples respectively, then*

$$\sum_{i=1}^{n}\sum_{j=1}^{m} p_i w_j \phi(a_i, c_j) \geq \sum_{i=1}^{n}\sum_{j=1}^{m} p_i w_j \phi(b_i, d_j) + c \sum_{i=1}^{n}\sum_{j=1}^{m} p_i w_j \left\{ (a_i - b_i)^2 + (c_j - d_j)^2 \right\}. \tag{2.25}$$

- *(ii) If $\boldsymbol{a}$ and $\boldsymbol{c}$ are increasing n and m-tuples respectively, then*

$$\sum_{i=1}^{n}\sum_{j=1}^{m} p_i w_j \phi(b_i, d_j) \geq \sum_{i=1}^{n}\sum_{j=1}^{m} p_i w_j \phi(a_i, c_j) + c \sum_{i=1}^{n}\sum_{j=1}^{m} p_i w_j \left\{ (a_i - b_i)^2 + (c_j - d_j)^2 \right\}. \tag{2.26}$$

Proof. For the proof, using Abel's transformation. Let

$$A_k = \sum_{i=1}^{k} p_i a_i, \;\; B_k = \sum_{i=1}^{k} p_i b_i, \;\; (k = 1, 2, ..., n); \;\; A_0 = B_0 = 0 \tag{2.27}$$

and

$$C_k = \sum_{j=1}^{k} w_j c_j, \;\; D_k = \sum_{j=1}^{k} w_j d_j, \;\; (k = 1, 2, ..., m); \;\; C_0 = D_0 = 0. \tag{2.28}$$

From (2.23) and (2.24), we have

$$A_n = B_n, \quad C_m = D_m. \tag{2.29}$$

Since ϕ is a strongly convex function on $\mathbf{I}_1 \times \mathbf{I}_2$, therefore ϕ is a co-ordinate strongly convex function on $\mathbf{I}_1 \times \mathbf{I}_2$. If $\mathbf{b}$ and $\mathbf{d}$ are decreasing n and $m-$tuples respectively, then $(t_1, t_2, ..., t_n)$ and $(s_1, s_2, ..., s_m)$ are decreasing n and $m-$tuples respectively, where t_i is the positive partial derivative of ϕ with respect to first variable at $b_i (i = 1, 2, ..., n)$ and s_j is the positive partial derivative of ϕ with respect to second variable at $d_j (j = 1, 2, ..., m)$. Using (2.8), we have

$$\begin{aligned}
&\sum_{i=1}^{n}\sum_{j=1}^{m} p_i w_j \phi(a_i, c_j) - \sum_{i=1}^{n}\sum_{j=1}^{m} p_i w_j \phi(b_i, d_j) \\
&\geq \sum_{i=1}^{n}\sum_{j=1}^{m} p_i w_j t_i (a_i - b_i) + \sum_{i=1}^{n}\sum_{j=1}^{m} p_i w_j s_j (c_j - d_j) + c\sum_{i=1}^{n}\sum_{j=1}^{m} p_i w_j \left\{(a_i - b_i)^2 + (c_j - d_j)^2\right\} \\
&= \sum_{j=1}^{m} w_j \Big[\sum_{i=1}^{n} t_i (p_i a_i - p_i b_i)\Big] + \sum_{i=1}^{n} p_i \Big[\sum_{j=1}^{m} s_j (w_j c_j - w_j d_j)\Big] \\
&\qquad + c\sum_{i=1}^{n}\sum_{j=1}^{m} p_i w_j \left\{(a_i - b_i)^2 + (c_j - d_j)^2\right\} \\
&= \sum_{j=1}^{m} w_j \Big[\sum_{i=1}^{n} t_i (A_i - A_{i-1} - B_i + B_{i-1})\Big] + \sum_{i=1}^{n} p_i \Big[\sum_{j=1}^{m} s_j (C_i - C_{i-1} - D_i - D_{i-1}\Big] \\
&\qquad + c\sum_{i=1}^{n}\sum_{j=1}^{m} p_i w_j \left\{(a_i - b_i)^2 + (c_j - d_j)^2\right\} \\
&= \sum_{j=1}^{m} w_j \Big[\sum_{i=1}^{n} t_i (A_i - B_i) - \sum_{i=1}^{n} t_i (A_{i-1} - B_{i-1})\Big] \\
&\quad + \sum_{i=1}^{n} p_i \Big[\sum_{j=1}^{m} s_j (C_i - D_i) - \sum_{j=1}^{m} s_j (C_{i-1} - D_{i-1})\Big] + c\sum_{i=1}^{n}\sum_{j=1}^{m} p_i w_j \left\{(a_i - b_i)^2 + (c_j - d_j)^2\right\} \\
&\ = \sum_{j=1}^{m} w_j \Big[\sum_{i=1}^{n-1} (t_i - t_{i+1})(A_i - B_i)\Big] + \sum_{i=1}^{n} p_i \Big[\sum_{j=1}^{m-1} (s_j - s_{j+1}(C_i - D_i)\Big] \\
&\qquad + c\sum_{i=1}^{n}\sum_{j=1}^{m} p_i w_j \left\{(a_i - b_i)^2 + (c_j - d_j)^2\right\}.
\end{aligned} \tag{2.30}$$

Since $(t_1, t_2, ..., t_n)$ and $(s_1, s_2, ..., s_m)$ are decreasing n and $m-$tuples respectively, therefore $t_i - t_{i+1} \geq 0$ $(i = 1, 2, ..., n-1)$ and $s_j - s_{j+1} \geq 0$ $(j = 1, 2, ..., m-1)$. Also from assumptions (2.21) and (2.22), we have $A_i - B_i \geq 0$ $(i = 1, 2, ..., n-1)$ and $C_j - D_j \geq 0$ $(j = 1, 2, ..., m-1)$. Thus

$$\sum_{j=1}^{m} w_j \Big[\sum_{i=1}^{n-1} (t_i - t_{i+1})(A_i - B_i)\Big] + \sum_{i=1}^{n} p_i \Big[\sum_{j=1}^{m-1} (s_j - s_{j+1}(C_i - D_i)\Big] \geq 0. \tag{2.31}$$

Using (2.31) in (2.30), we get inequality (2.25).

Similarly, we can prove inequality (2.26) for the remaining cases. □

Now, to present another result for strongly convex functions and for arbitrary monotonic tuples.

Theorem 2.4. *Let all the assumptions of Theorem 2.2 hold. If* **b** *and* **a-b** *are monotonic n-tuples in the same sense, and* **c** *and* **c-d** *are monotonic m-tuples in the same sense. Then*

$$\begin{aligned}&\sum_{i=1}^{n}\sum_{j=1}^{m}p_iw_j\phi(a_i,c_j)-\sum_{i=1}^{n}\sum_{j=1}^{m}p_iw_j\phi(b_i,d_j)\\&\geq\frac{1}{P_n}\sum_{j=1}^{m}\sum_{i=1}^{n}w_jp_it_i\sum_{i=1}^{n}p_i(a_i-b_i)+\frac{1}{W_m}\sum_{i=1}^{n}\sum_{j=1}^{m}p_iw_js_j\sum_{j=1}^{m}w_j(c_j-d_j)\\&+c\sum_{i=1}^{n}\sum_{j=1}^{m}p_iw_j\left\{(a_i-b_i)^2+(c_j-d_j)^2\right\},\end{aligned}\tag{2.32}$$

where t_i is the positive partial derivative of ϕ with respect to first variable at b_i (i=1,2,...,n) and s_j is the partial positive derivative of ϕ with respect to second variable at d_j (j=1,2,...,m).

Proof. It follows from the proof of Proposition 2.1 that $(t_1,t_2,...,t_n)$ is an increasing n-tuple. Now if **b** and **a-b** are monotonic increasing in mean by the report of **p**, then applying Chebyshev's inequality to first term on right hand side of (2.8), we have

$$\begin{aligned}\sum_{i=1}^{n}\sum_{j=1}^{m}p_iw_jt_i(a_i-b_i)&=\sum_{j=1}^{m}w_j\left[\sum_{i=1}^{n}p_it_i(a_i-b_i)\right]\\&\geq\sum_{j=1}^{m}w_j\left[\frac{1}{P_n}\sum_{i=1}^{n}p_it_i\sum_{i=1}^{n}p_i(a_i-b_i)\right]\\&=\frac{1}{P_n}\sum_{j=1}^{m}\sum_{i=1}^{n}w_jp_it_i\sum_{i=1}^{n}p_i(a_i-b_i).\end{aligned}\tag{2.33}$$

Similarly, we have

$$\sum_{i=1}^{n}\sum_{j=1}^{m}p_iw_js_j(c_j-d_j)\geq\frac{1}{W_m}\sum_{i=1}^{n}\sum_{j=1}^{m}p_iw_js_j\sum_{j=1}^{m}w_j(c_j-d_j).\tag{2.34}$$

Using (2.33) and (2.34) in (2.8), we get (2.32).

Similarly, we can prove inequality (2.32) in the remaining cases. □

Corollary 2.1. *Assume that all the hypotheses of Theorem 2.4 hold. Additionally, if*

$$\sum_{i=1}^{n}a_ip_i=\sum_{i=1}^{n}b_ip_i\tag{2.35}$$

and

$$\sum_{j=1}^{m}c_jw_j=\sum_{j=1}^{m}d_jw_j.\tag{2.36}$$

Then

$$\begin{aligned}&\sum_{i=1}^{n}\sum_{j=1}^{m}p_iw_j\phi(a_i,c_j)\\&\geq\sum_{i=1}^{n}\sum_{j=1}^{m}p_iw_j\phi(b_i,d_j)+c\sum_{i=1}^{n}\sum_{j=1}^{m}p_iw_j\left\{(a_i-b_i)^2+(c_j-d_j)^2\right\}.\end{aligned}\tag{2.37}$$

Proof. Using (2.35) and (2.36) on the right hand side of (2.32), we have (2.37). □

In the following result, we obtain majorization inequality by using increasing strongly convex function.

Corollary 2.2. *Let all the assumptions of Theorem 2.4 hold. Moreover, if* $\phi:\mathbf{I_1}\times\mathbf{I_2}\longrightarrow\mathbb{R}$ *is an increasing strongly convex function, such that*

$$\sum_{i=1}^{n}a_ip_i\geq\sum_{i=1}^{n}b_ip_i\tag{2.38}$$

and

$$\sum_{j=1}^{m}c_jw_j\geq\sum_{j=1}^{m}d_jw_j.\tag{2.39}$$

Then inequality (2.37) *holds.*

Proof. Since ϕ is an increasing function on $\mathbf{I_1}\times\mathbf{I_2}$, therefore $t_i\geq 0$ $(i=1,2,...,n)$, $s_j\geq 0$ $(j=1,2,...,n)$, where t_i is positive partial derivative of ϕ with respect to first variable at b_i $(i=1,2,...,n)$ and s_j is positive partial derivative of ϕ with respect to second variable at d_j $(j=1,2,...,m)$, thus

$$\sum_{i=1}^{n}p_it_i\geq 0\tag{2.40}$$

and

$$\sum_{j=1}^{m}w_js_j\geq 0\tag{2.41}$$

Using (2.38), (2.39), (2.40)and (2.41) on the right hand side of (2.32), we have (2.37). □

The following lemma is given in [29].

Lemma 2.1. *Let* $\boldsymbol{v}$ *be a positive real* $n-$*tuple. If* $\boldsymbol{x}$ *is an increasing real* $n-$*tuple, then*

$$\sum_{i=1}^{k}x_iv_i\sum_{i=1}^{n}v_i\leq\sum_{i=1}^{n}x_iv_i\sum_{i=1}^{k}v_i,\quad k=1,2,...,n.\tag{2.42}$$

If $\boldsymbol{x}$ *is a decreasing real* $n-$*tuple, then the reverse inequality holds in* (2.42).

If $\mathbf{a}=(a_1,a_2,...,a_n)$ and $\mathbf{b}=(b_1,b_2,...,b_n)$, then $\frac{\mathbf{a}}{\mathbf{b}}=(\frac{a_1}{b_1},\frac{a_2}{b_2},...,\frac{a_n}{b_n})$.

Theorem 2.5. *Let* $\mathbf{I_1}$, $\mathbf{I_2}$ *be any two intervals in* $\mathbb{R}$ *and* $\phi:\mathbf{I_1}\times\mathbf{I_2}\longrightarrow\mathbb{R}$ *be a strongly convex functions. Also let* $\boldsymbol{a}=(a_1,a_2,...,a_n)$ *and* $\boldsymbol{b}=(b_1,b_2,...,b_n)$ *be positive* n*-tuples such that* $a_i,b_i\in\mathbf{I_1}$ $(i=1,2,...,n)$, $\boldsymbol{c}=(c_1,c_2,...,c_m)$ *and* $\boldsymbol{d}=(d_1,d_2,...,d_m)$ *be positive* m*-tuples* *such that* $c_j, d_j \in \mathbf{I_2}$ $(j = 1, 2, ..., m)$, *and* $\boldsymbol{p} = (p_1, p_2, ..., p_n)$ *and* $\boldsymbol{w} = (w_1, w_2, ..., w_m)$ *be any positive real* n *and* m*-tuples respectively.*

If $\boldsymbol{a}/\mathbf{b}$ *and* $\mathbf{c}/\boldsymbol{d}$ *are decreasing* n *and* m*-tuples respectively, then the following statements are true:*

- *(i) If* $\boldsymbol{a}$ *is increasing* n*-tuple and* $\mathbf{c}$ *is increasing* m*-tuple, then*

$$\sum_{i=1}^{n}\sum_{j=1}^{m} p_i w_j \phi\left(\frac{b_i}{\sum_{i=1}^{n} b_i p_i}, \frac{d_j}{\sum_{j=1}^{m} w_j d_j}\right)$$
$$\geq \sum_{i=1}^{n}\sum_{j=1}^{m} p_i w_j \left(\frac{a_i}{\sum_{i=1}^{n} p_i a_i}, \frac{c_j}{\sum_{j=1}^{m} w_j c_j}\right) \tag{2.43}$$
$$+ c\sum_{i=1}^{n}\sum_{j=1}^{m} p_i w_j \left\{\left(\frac{a_i}{\sum_{i=1}^{n} p_i a_i} - \frac{b_i}{\sum_{i=1}^{n} b_i p_i}\right)^2 + \left(\frac{c_j}{\sum_{j=1}^{m} w_j c_j} - \frac{d_j}{\sum_{j=1}^{m} w_j d_j}\right)^2\right\}$$

- *(ii) If* $\mathbf{b}$ *is an decreasing* n*-tuple and* $\mathbf{d}$ *is an decreasing* m*-tuple, then*

$$\sum_{i=1}^{n}\sum_{j=1}^{m} p_i w_j \phi\left(\frac{a_i}{\sum_{i=1}^{n} p_i a_i}, \frac{c_j}{\sum_{j=1}^{m} w_j c_j}\right)$$
$$\geq \sum_{i=1}^{n}\sum_{j=1}^{m} p_i w_j \phi\left(\frac{b_i}{\sum_{i=1}^{n} b_i p_i}, \frac{d_j}{\sum_{j=1}^{m} w_j d_j}\right) \tag{2.44}$$
$$+ c\sum_{i=1}^{n}\sum_{j=1}^{m} p_i w_j \left\{\left(\frac{a_i}{\sum_{i=1}^{n} p_i a_i} - \frac{b_i}{\sum_{i=1}^{n} b_i p_i}\right)^2 + \left(\frac{c_j}{\sum_{j=1}^{m} w_j c_j} - \frac{d_j}{\sum_{j=1}^{m} w_j d_j}\right)^2\right\}$$

If $\boldsymbol{a}/\mathbf{b}$ *and* $\mathbf{c}/\boldsymbol{d}$ *are increasing* n *and* m*-tuples respectively, then we have the following statements:*

- *(iii) If* $\mathbf{b}$ *is increasing* n*-tuple and* $\mathbf{d}$ *is increasing* m*-tuple, then* (2.44) *holds.*
- *(iv) If* $\boldsymbol{a}$ *is an decreasing* n*-tuple and* $\mathbf{c}$ *is an decreasing* m*-tuple, then* (2.43) *holds.*

Proof. (i) Let $\mathbf{a}/\mathbf{b}$ and $\mathbf{c}/\mathbf{d}$ are decreasing n and $m-tuples$ respectively. Then using Lemma 2.1 with

$$\mathbf{x} = \mathbf{a}/\mathbf{b}, \quad \mathbf{v} = \mathbf{pb}$$

we obtain

$$\sum_{i=1}^{k} a_i p_i \sum_{i=1}^{n} p_i b_i \geq \sum_{i=1}^{n} p_i a_i \sum_{i=1}^{k} p_i b_i, \quad k = 1, 2, ..., n.$$

That is

$$\sum_{i=1}^{k} p_i \left(\frac{a_i}{\sum_{i=1}^{n} p_i a_i}\right) \geq \sum_{i=1}^{k} p_i \left(\frac{b_i}{\sum_{i=1}^{n} p_i b_i}\right), \quad k = 1, 2, ..., n. \tag{2.45}$$

Again using Lemma 2.1 with

$$\mathbf{x} = \mathbf{c}/\mathbf{d}, \quad \mathbf{v} = \mathbf{dw}$$

we obtain

$$\sum_{i=1}^{k} c_j w_j \sum_{i=1}^{n} d_j w_j \geq \sum_{i=1}^{n} w_j c_j \sum_{i=1}^{k} w_j d_j, \quad k = 1, 2, ..., m.$$

That is

$$\sum_{j=1}^{k} w_j\left(\frac{c_j}{\sum_{j=1}^{m} w_j c_j}\right) \geq \sum_{j=1}^{k} w_j\left(\frac{d_j}{\sum_{j=1}^{m} w_j d_j}\right), \quad k = 1, 2, ..., m. \tag{2.46}$$

From (2.45) and (2.46), we have

$$\sum_{i=1}^{n} p_i\left(\frac{a_i}{\sum_{i=1}^{n} p_i a_i}\right) = \sum_{i=1}^{n} p_i\left(\frac{b_i}{\sum_{i=1}^{n} p_i b_i}\right) \tag{2.47}$$

and

$$\sum_{j=1}^{n} w_j\left(\frac{c_j}{\sum_{j=1}^{m} w_j c_j}\right) = \sum_{j=1}^{n} w_j\left(\frac{d_j}{\sum_{j=1}^{m} w_j d_j}\right). \tag{2.48}$$

If $\mathbf{a}$ and $\mathbf{c}$ are increasing. Then using Theorem 2.3(ii) and the conditions $(2.45) - (2.48)$ we obtain inequality (2.43).

Similarly, we can prove the remaining cases. □

Theorem 2.6. *Let $\phi : [0, \infty)\times[0, \infty) \longrightarrow \mathbb{R}$ be a strongly convex function, $\boldsymbol{a} = (a_1, a_2, ..., a_n)$ and $\boldsymbol{b} = (b_1, b_2, ..., b_n)$ be positive n-tuples, $\boldsymbol{c} = (c_1, c_2, ..., c_m)$ and $\boldsymbol{d} = (d_1, d_2, ..., d_m)$ be positive m-tuples, and $\boldsymbol{p} = (p_1, p_2, ..., p_n)$ and $\boldsymbol{w} = (w_1, w_2, ..., w_m)$ be any positive real n and m-tuples respectively. Then the following statements are true:*

- *(i) If $\boldsymbol{a}$ is increasing concave n-tuple and $\boldsymbol{c}$ is increasing concave m-tuple, then*

$$\begin{aligned}&\sum_{i=1}^{n}\sum_{j=1}^{m} p_i w_j \phi\left(\frac{i-1}{\sum_{i=1}^{n} p_i(i-1)}, \frac{j-1}{\sum_{j=1}^{m} w_j(j-1)}\right)\\ &\geq \sum_{i=1}^{n}\sum_{j=1}^{m} p_i w_j \phi\left(\frac{a_i}{\sum_{i=1}^{n} p_i a_i}, \frac{c_j}{\sum_{j=1}^{m} w_j c_j}\right)\\ &+ c\sum_{i=1}^{n}\sum_{j=1}^{m} p_i w_j \left\{\left(\frac{a_i}{\sum_{i=1}^{n} p_i a_i} - \frac{i-1}{\sum_{i=1}^{n} p_i(i-1)}\right)^2 + \left(\frac{c_j}{\sum_{j=1}^{m} w_j c_j} - \frac{j-1}{\sum_{j=1}^{m} w_j(j-1)}\right)^2\right\}\end{aligned} \tag{2.49}$$

- *(ii) If $\boldsymbol{a}$ is increasing convex n-tuple with $a_1 = 0$ and $\boldsymbol{c}$ is increasing convex m-tuple with $c_1 = 0$, then*

$$\begin{aligned}&\sum_{i=1}^{n}\sum_{j=1}^{m} p_i w_j \phi\left(\frac{a_i}{\sum_{i=1}^{n} p_i a_i}, \frac{c_j}{\sum_{j=1}^{m} w_j c_j}\right)\\ &\geq \sum_{i=1}^{n}\sum_{j=1}^{m} p_i w_j \phi\left(\frac{i-1}{\sum_{i=1}^{n} p_i(i-1)}, \frac{j-1}{\sum_{j=1}^{m} w_j(j-1)}\right)\\ &+ c\sum_{i=1}^{n}\sum_{j=1}^{m} p_i w_j \left\{\left(\frac{a_i}{\sum_{i=1}^{n} p_i a_i} - \frac{i-1}{\sum_{i=1}^{n} p_i(i-1)}\right)^2 + \left(\frac{c_j}{\sum_{j=1}^{m} w_j c_j} - \frac{j-1}{\sum_{j=1}^{m} w_j(j-1)}\right)^2\right\}\end{aligned} \tag{2.50}$$

- *(iii) If* $\boldsymbol{a}$ *is decreasing concave n-tuple and* $\boldsymbol{c}$ *is decreasing concave m-tuple, then*

$$\sum_{i=1}^{n}\sum_{j=1}^{m} p_i w_j \phi\left(\frac{n-i}{\sum_{i=1}^{n} p_i(n-i)}, \frac{m-j}{\sum_{j=1}^{m} w_j(m-j)}\right)$$
$$\geq \sum_{i=1}^{n}\sum_{j=1}^{m} p_i w_j \phi\left(\frac{a_i}{\sum_{i=1}^{n} p_i a_i}, \frac{c_j}{\sum_{j=1}^{m} w_j c_j}\right)$$
$$+ c\sum_{i=1}^{n}\sum_{j=1}^{m} p_i w_j \left\{\left(\frac{a_i}{\sum_{i=1}^{n} p_i a_i} - \frac{n-i}{\sum_{i=1}^{n} p_i(n-i)}\right)^2 + \left(\frac{c_j}{\sum_{j=1}^{m} w_j c_j} - \frac{m-j}{\sum_{j=1}^{m} w_j(m-j)}\right)^2\right\} \tag{2.51}$$

- *(iv) If* $\boldsymbol{a}$ *is decreasing convex n-tuple with* $a_n = 0$ *and* $\boldsymbol{c}$ *is decreasing convex m-tuple with* $c_m = 0$*, then*

$$\sum_{i=1}^{n}\sum_{j=1}^{m} p_i w_j \phi\left(\frac{a_i}{\sum_{i=1}^{n} p_i a_i}, \frac{c_j}{\sum_{j=1}^{m} w_j c_j}\right)$$
$$\geq \sum_{i=1}^{n}\sum_{j=1}^{m} p_i w_j \phi\left(\frac{n-i}{\sum_{i=1}^{n} p_i(n-i)}, \frac{m-j}{\sum_{j=1}^{m} w_j(m-j)}\right)$$
$$+ c\sum_{i=1}^{n}\sum_{j=1}^{m} p_i w_j \left\{\left(\frac{a_i}{\sum_{i=1}^{n} p_i a_i} - \frac{n-i}{\sum_{i=1}^{n} p_i(n-i)}\right)^2 + \left(\frac{c_j}{\sum_{j=1}^{m} w_j c_j} - \frac{m-j}{\sum_{j=1}^{m} w_j(m-j)}\right)^2\right\} \tag{2.52}$$

Proof. Let $b = (b_1, b_2, ..., b_n)$ and $d = (d_1, d_2, ..., d_m)$ be respectively the n- and m-tuples such that $b_1 = \epsilon < a_1/a_2, d_1 = \delta < c_1/c_2, b_i = i-1$ for $i = 2, 3, ..., n$, and $d_j = j-1$ for $j = 2, 3, ..., m$. Then a/b and c/d are decreasing n- and m-tuples, respectively. It follows from Theorem 2.5 that

$$p_1 w_1 \phi\left(\frac{\epsilon}{\epsilon p_1 + \sum_{i=2}^{n}(i-1)p_i}, \frac{\delta}{w_1\delta + \sum_{j=2}^{m}(j-1)w_j}\right)$$
$$+ \sum_{i=2}^{n}\sum_{j=2}^{m} p_i w_j \phi\left(\frac{i-1}{\epsilon p_1 + \sum_{i=2}^{n}(i-1)p_i}, \frac{j-1}{w_1\delta + \sum_{j=2}^{m}(j-1)w_j}\right)$$
$$\geq \sum_{i=1}^{n}\sum_{j=1}^{m} p_i w_j \phi\left(\frac{a_i}{\sum_{i=1}^{n} p_i a_i}, \frac{c_j}{\sum_{j=1}^{m} w_j c_j}\right) + c p_1 w_1 \left\{\left(\frac{a_1}{\sum_{i=1}^{n} p_i a_i} - \frac{\epsilon}{\epsilon p_1 + \sum_{i=2}^{n} p_i(i-1)}\right)^2\right.$$
$$\left. + \left(\frac{c_1}{\sum_{j=1}^{m} w_j c_j} - \frac{\delta}{\delta w_1 + \sum_{j=2}^{m} w_j(j-1)}\right)^2\right\}$$
$$+ c\sum_{i=2}^{n}\sum_{j=2}^{m} p_i w_j \left\{\left(\frac{a_i}{\sum_{i=1}^{n} p_i a_i} - \frac{i-1}{\epsilon p_1 + \sum_{i=2}^{n} p_i(i-1)}\right)^2\right.$$
$$\left. + \left(\frac{c_j}{\sum_{j=1}^{m} w_j c_j} - \frac{j-1}{\delta w_1 + \sum_{j=2}^{m} w_j(m-j)}\right)^2\right\}$$

Taking $\epsilon \to 0$ and $\delta \to 0$,we obtain

$$
\begin{aligned}
&p_1 w_1 \phi(0,0) + \sum_{i=2}^{n}\sum_{j=2}^{m} p_i w_j \phi\left(\frac{i-1}{\epsilon p_1 + \sum_{i=2}^{n}(i-1)p_i}, \frac{j-1}{w_1\delta + \sum_{j=2}^{m}(j-1)w_j}\right)\\
&\quad \geq \sum_{i=1}^{n}\sum_{j=1}^{m} p_i w_j \phi\left(\frac{a_i}{\sum_{i=1}^{n} p_i a_i}, \frac{c_j}{\sum_{j=1}^{m} w_j c_j}\right) + c p_1 w_1 \left\{\left(\frac{a_1}{\sum_{i=1}^{n} p_i a_i}\right)^2 + \left(\frac{c_1}{\sum_{j=1}^{m} w_j c_j} w_1\right)^2\right\}\\
&\quad + c \sum_{i=2}^{n}\sum_{j=2}^{m} p_i w_j \left\{\left(\frac{a_i}{\sum_{i=1}^{n} p_i a_i} - \frac{i-1}{\epsilon p_1 + \sum_{i=2}^{n} p_i (i-1)}\right)^2\right.\\
&\qquad \left. + \left(\frac{c_j}{\sum_{j=1}^{m} w_j c_j} - \frac{j-1}{\delta w_1 + \sum_{j=2}^{m} w_j (j-1)}\right)^2\right\}\\
&= \sum_{i=1}^{n}\sum_{j=1}^{m} p_i w_j \phi\left(\frac{a_i}{\sum_{i=1}^{n} p_i a_i}, \frac{c_j}{\sum_{j=1}^{m} w_j c_j}\right)\\
&\quad + c \sum_{i=1}^{n}\sum_{j=1}^{m} p_i w_j \left\{\left(\frac{a_i}{\sum_{i=1}^{n} p_i a_i} - \frac{i-1}{\sum_{i=1}^{n} p_i (i-1)}\right)^2 + \left(\frac{c_j}{\sum_{j=1}^{m} w_j c_j} - \frac{j-1}{\sum_{j=1}^{m} w_j (j-1)}\right)^2\right\}
\end{aligned}
$$

This proves(2.49).

Similarly we can use Theorem (2.5) to prove the required result for the remaining cases. □

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