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FABER POLYNOMIAL COEFFICIENT ESTIMATION OF SUBCLASS OF BI-SUBORDINATE UNIVALENT FUNCTIONS

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Abstract. In this paper, a comprehensive subclass of bi-univalent functions class are introduced and investigated. Using the Faber polynomials, estimation of the coefficients $|a_n|$ and certain Fekete-Szegö inequality of Maclaurin expansion of functions in this subclass are concluded. Finally, some earlier results are pointed out and improved.

1. Introduction

Let $\mathcal{A}$ denote the class of analytic functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

normalized by the conditions $f(0) = 0$ and $f^{'}(0) = 1$ defined in the open unit disk

$$U = \{z \in \mathbb{C} : |z| < 1\}.$$

Let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of all functions of the form (1.1) which are univalent in U. Let φ be an analytic univalent function in U with positive real part and $\varphi(U)$ be symmetric with respect to the real axis, starlike with respect to $\varphi(0) = 1$ and $\varphi^{'}(0) > 0$. Ma and Minda [17] gave a unified presentation of various subclasses of starlike and convex functions by introducing the classes $\mathfrak{S}^*(\varphi)$ and $\mathfrak{K}(\varphi)$ of functions $f \in \mathcal{S}$ satisfying $(zf^{'}(z)/f(z)) \prec \varphi(z)$ and $1 + (zf^{''}(z)/f^{'}(z)) \prec \varphi(z)$ respectively, which includes several well-known classes as special case. For example, when $\varphi(z) = (1 + Az)/(1 + Bz)$ with a condition $(-1 \leq B < A \leq 1)$, the classes $\mathfrak{S}^*(\varphi)$ and $\mathfrak{K}(\varphi)$ converted to the class $\mathcal{S}^*[A, B]$ and $\mathcal{K}[A, B]$, respectively, introduced by Janowski [15]. Although, for a special choose of the value of $A = 1 - 2\beta,\ B = -1\ (0 \leq \beta < 1)$, the classes $\mathcal{S}^*[A, B]$ and $\mathcal{K}[A, B]$ reduced to the classes $\mathcal{S}^*(\beta)$ and $\mathcal{K}(\beta)$, respectively, which are the class of starlike and convex functions of order β. For anther choose of the function $\varphi(z) = ((1 + z)/(1 - z))^{\alpha}$, we obtain the classes

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$\mathcal{S}^*_\alpha$ and $\mathcal{K}_\alpha$ which are the class of strongly starlike and strongly convex functions of order α $(0 < \alpha \leq 1)$.

The Koebe one quarter theorem [8] ensures that the image of U under every univalent function $f \in \mathcal{S}$ contains a disk of radius $\frac{1}{4}$. Thus every univalent function f has an inverse f^{-1} satisfying

$$f^{-1}(f(z)) = z, \quad (z \in U) \text{ and } f(f^{-1}(w)) = w \ (|w| < r_0(f), r_0(f) \leq \frac{1}{4}).$$

A function $f \in \mathcal{S}$ is said to be bi-univalent in U if both f and f^{-1} are univalent in U. Let Σ denote the class of all bi-univalent functions defined in the unit disk U. Since $f \in \Sigma$ has the Maclaurin series expansion given by (1.1), a simple calculation shows that its inverse $g = f^{-1}$ has the series expansion

$$\begin{aligned} g(w) &= f^{-1}(w) \\ &= w - a_2 w^2 + (2a_2^2 - a_3)w^3 - .... \end{aligned}$$

Examples of functions in the class Σ are

$$\frac{z}{1-z}, \quad -\log(1-z) \quad \text{and} \quad \frac{1}{2}\log\left(\frac{1+z}{1-z}\right),$$

and so on. However, the familiar Koebe function is not a member of Σ.
Other common examples of functions in $\mathcal{S}$ such as

$$z - \frac{z^2}{2} \quad \text{and} \quad \frac{z}{1-z^2}$$

are also not members of Σ (see [20]).

Many papers concerning bi-univalent functions have been published recently (for mentioned but a few, [5,6,9,11]). A function $f \in \Sigma$ is in the class $\mathcal{S}^*_\Sigma(\beta)$ of bi-starlike function of order $\beta (0 \leq \beta < 1)$, or $\mathcal{K}_\Sigma(\beta)$ of bi-convex function of order β if both f and f^{-1} are respectively starlike or convex functions of order β. For $0 < \alpha \leq 1$, the function $f \in \Sigma$ is strongly bi-starlike function of order α if both the functions f and f^{-1} are strongly starlike functions of order α. The class of all such functions is denoted by $\mathcal{S}^*_{\Sigma,\alpha}$. These classes were introduced by Brannan and Taha [5]. They obtained estimates on the initial coefficients $|a_2|$ and $|a_3|$ for functions in these classes. The research into Σ was started by Lewin [16]. He focused on problems connected with coefficients and showed that $|a_2| < 1.51$. Subsequently, Brannan and Clunie [4] conjectured that $|a_2| < \sqrt{2}$. Netanyahu [19] concluded that $\max |a_2| = \frac{4}{3}$.

The coefficient estimate problem for each of the following Taylor Maclaurin coefficients $|a_n|$, $n \in \{2, 3, \cdots\}$ is presumably still an open problem. This is because the bi-univalency requirement makes the behavior of the coefficients of the function f and f^{-1} unpredictable. The Faber polynomials play an important role in various areas of mathematical sciences, especially in geometric function theory. The recent publications [12,13] applying the Faber polynomial expansions to meromorphic bi-univalent functions motivated us to apply this technique to classes of analytic bi-univalent functions. In the literature, there are only a few works determining the general coefficient bounds $|a_n|$ for the analytic bi-univalent functions given by (1.1) using Faber polynomial expansions.

In this present work, we use the Faber polynomials in obtaining bounds of Maclaurin coefficients $|a_n|$, $n \in \mathbb{N}$ 1 and bounds for the Fekete-Szegö functional $|a_3 - 2a_2^2|$ of a new defined subclass of Σ to generalize some earlier results.

2. Construction of the subclass $\mathcal{H}_\Sigma(\tau, \lambda, \delta; \varphi)$

Throughout this section, let us assume that φ be an analytic function with positive real part in the unit disc U satisfying $\varphi(0) = 1$, $\varphi'(0) > 0$ and $\varphi(U)$ is symmetric with respect to the real axis. Such a function has a series expansion of the form

$$\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots \qquad (B_1 > 0). \tag{2.1}$$

where $B_n \in \mathbb{R}$, for all $n = 2, 3, ...$

Using the Faber polynomial [1, 2] expansion of the functions $f \in \Sigma$ of the form (1.1), the inverse function $g = f^{-1}$ may be expressed as

$$g(w) = f^{-1}(w) = w + \sum_{n=2}^{\infty} A_n w^n, \tag{2.2}$$

where

$$A_n = \frac{1}{n}\mathcal{K}_{n-1}^{-n}(a_2, a_3, ..., a_n). \tag{2.3}$$

Now, for any $p \in \mathbb{Z} := \{0, \pm 1, \pm 2, \cdots\}$, the expansion of $\mathcal{K}_n^p$ is given by

$$\mathcal{K}_n^p = pa_n + \frac{p!}{(p-2)!2!}D_n^2 + \frac{p!}{(p-3)!3!}D_n^3 + \cdots + \frac{p!}{(p-n)!n!}D_n^n, \tag{2.4}$$

where

$$\begin{aligned} D_n^m &= D_n^m(a_1, a_2, ..., a_n), \\ &= \sum_{n=2}^{\infty} \frac{m!}{\mu_1!\mu_2!\mu_3!\cdots\mu_n!} a_1^{\mu_1} a_2^{\mu_2} a_3^{\mu_3} \cdots a_n^{\mu_n}, \end{aligned} \tag{2.5}$$

while $a_1 = 1$ and the sum is taken over all non-negative integers $\mu_1, \mu_2, \mu_3, ..., \mu_n$ satisfying

$$\mu_1 + \mu_2 + \mu_3 + \cdots + \mu_n = m, \qquad \mu_1 + 2\mu_2 + \cdots + n\mu_n = n.$$

It is observed that

$$D_n^n(a_1, a_3, ..., a_n) = a_1^n.$$

Thus, from equation (2.4) together with (2.5) we get an expression of $\mathcal{K}_{n-1}^{-n}$ as

$$\begin{aligned} \mathcal{K}_{n-1}^{-n}(a_2, a_3, ..., a_n) &= \frac{(-n)!}{(-2n+1)!(n-1)!}a_2^{n-1} + \frac{(-n)!}{(2(-n+1))!(n-3)!}a_2^{n-3}a_3 \\ &+ \frac{(-n)!}{(-2n+3))!(n-4)!}a_2^{n-4}a_4 \\ &+ \frac{(-n)!}{(2(-n+2))!(n-5)!}a_2^{n-5}\left(a_5 + (-n+2)a_3^2\right) \\ &+ \frac{(-n)!}{(-2n+5))!(n-6)!}a_2^{n-6}\left(a_6 + (-2n+5)a_3 a_4\right) \\ &+ \sum_{j \geq 7} a_2^{n-j} V_j, \end{aligned}$$

where such expressions as $(-n)!$ are to be interpreted by

$$(-n)! := \Gamma(1-n) = (-n)(-n-1)(-n-2)\cdots \ (\ n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\} \ ),$$

and V_j $(7 \leq j \leq n)$ is a homogeneous polynomial in the variables $a_2, a_3, ..., a_n$. In particular, in case of $n = 2, 3, 4$ the expression of $\mathcal{K}_{n-1}^{-n}$ is reduced to

$$\mathcal{K}_1^{-2} = -2a_2, \quad \mathcal{K}_2^{-3} = 3(2a_2^2 - a_3), \quad \mathcal{K}_3^{-4} = -4(5a_2^3 - 5a_2a_3 + a_4).$$

Definition 2.1. Let $\lambda \geq 1, \tau \in \mathbb{C}^* = \mathbb{C} - \{0\}$, $0 \leq \delta \leq 1$ and $f, g \in \Sigma$ given by (1.1) and (2.2) respectively, then f is said to be in the class $\mathcal{H}_\Sigma(\tau, \lambda, \delta; \varphi)$ if

$$1 + \frac{1}{\tau}\left((1-\lambda)\frac{f(z)}{z} + \lambda f^{'}(z) + \delta z f^{''}(z) - 1\right) \prec \varphi(z), \tag{2.6}$$

and

$$1 + \frac{1}{\tau}\left((1-\lambda)\frac{g(w)}{w} + \lambda g^{'}(w) + \delta z g^{''}(w) - 1\right) \prec \varphi(w), \tag{2.7}$$

where $z, w \in U$ and $\varphi(z)$ is given by (2.1).

Remark 2.1. For special choices of the parameters λ, τ, δ and the function $\varphi(z)$, the class $\mathcal{H}_\Sigma(\tau, \lambda, \delta; \varphi)$ reduced to the following subclasses:

1. $\mathcal{H}_\Sigma(\tau, 1, \gamma; \varphi) = \Sigma(\tau, \gamma, \varphi)$ which introduced by A.E. Tudor [23] and recently studied by H.M. Srivastava and Deepak Bansal [22].
2. $\mathcal{H}_\Sigma(1, 1, 0; \varphi) = \mathcal{H}_\sigma(\varphi)$ which defined and studied by Rosihan M. Ali et al. [3].
3. $\mathcal{H}_\Sigma\left(1, 1, \beta; \left(\frac{1+z}{1-z}\right)^\alpha\right) = \mathcal{H}_\Sigma(\alpha, \beta)$ which introduced by B.A. Frasin [11].
4. $\mathcal{H}_\Sigma\left(1, 1, 0; \left(\frac{1+z}{1-z}\right)^\alpha\right) = \mathcal{H}_\Sigma^\alpha$ which introduced by H.M. Srivastava et al. [20].
5. $\mathcal{H}_\Sigma\left(1, \lambda, 0; \left(\frac{1+z}{1-z}\right)^\alpha\right) = \mathcal{B}_\Sigma(\alpha, \lambda)$ which is introduced by B.A. Frasin and M.K. Aouf [10], and recently studied by H.M. Srivastava et al. [21].
6. $\mathcal{H}_\Sigma\left(1-\gamma, 1, \beta; \frac{1+z}{1-z}\right) = \mathcal{H}_\Sigma(\gamma, \beta)$ which introduced by B.A. Frasin [11].
7. $\mathcal{H}_\Sigma\left(1-\alpha, \lambda, \delta; \frac{1+z}{1-z}\right) = \mathcal{N}_\Sigma(\alpha, \lambda, \delta)$ which introduced by S. Bulut [6].
8. $\mathcal{H}_\Sigma\left(1-\beta, 1, 0; \frac{1+z}{1-z}\right) = \mathcal{H}_\Sigma(\beta)$ which introduced by H.M. Srivastava et al. [20].
9. $\mathcal{H}_\Sigma\left(1-\beta, \lambda, 0; \frac{1+z}{1-z}\right) = \mathcal{B}_\Sigma(\beta, \lambda)$ which introduced by B.A. Frasin and M.A. Aouf [10] and recently studied by J.M. Jahangiri and S.G. Hamidi [14].
10. $\mathcal{H}_\Sigma\left(\tau, 1, \gamma; \frac{1+Az}{1+Bz}\right) = \mathcal{R}_{\gamma,\sigma}^\tau(A, B)$ which introduced by A.E. Tudor [23].

Lemma 2.1. [18] *Let $u(z)$ be analytic function in the unit disc $\mathbb{U}$ with $u(0) = 0$ and $|u(z)| < 1$ for all $z \in U$ with the power series expansion*

$$u(z) = \sum_{n=1}^{\infty} c_n z^n \ ,$$

then $|c_n| \leq 1$ for all $n = 1, 2, 3, ...$ Furthermore, $|c_n| = 1$ for some $n = 1, 2, 3, ...$ if and only if

$$u(z) = e^{i\theta} z^n, \qquad \theta \in \mathbb{R}.$$

Lemma 2.2. [7] *Let the function* $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$ *be so that* $\Re(p(z)) > 0$ *for* $z \in U$. *Then for* $-\infty < \alpha < \infty$,

$$\left|p_2 - \alpha p_1^2\right| \leq \begin{cases} 2 - \alpha|p_1|^2 & ; \ \alpha < \frac{1}{2} \\ 2 - (1-\alpha)|p_1|^2 & ; \ \alpha \geq \frac{1}{2} \end{cases}. \tag{2.8}$$

Let $\varphi(z) = \sum_{n=1}^{\infty} a_n z^n$ *be a Schwarz function so that* $|\varphi(z)| < 1$, $z \in U$. *Set* $p(z) = \frac{1+\varphi(z)}{1-\varphi(z)}$ *where* $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$ *is so that* $\Re(p(z)) > 0$ *for* $z \in U$. *Comparing the corresponding coefficients of powers of* z *yields* $p_1 = 2\varphi_1$ *and* $p_2 = 2(\varphi_2 + \varphi_1^2)$. *Now, substituting for* p_1 *and* p_2 *and letting* $\eta = 1 - 2\alpha$ *in* (2.8), *we obtain*

$$\left|\varphi_2 + \eta\varphi_1^2\right| \leq \begin{cases} 1 - (1-\eta)|\varphi_1|^2 & ; \ \eta > 0 \\ 1 - (1+\eta)|\varphi_1|^2 & ; \ \eta < 0 \end{cases}. \tag{2.9}$$

2.1. Coefficient bounds of members of $\mathcal{H}_\Sigma(\tau, \lambda, \delta; \varphi)$.

Unless otherwise mentioned, let us assume in the reminder of this section that $z \in U$, $\lambda \geq 1$, $0 \leq \delta \leq 1$ and $\tau \in \mathbb{C} - \{0\}$.

Theorem 2.1. *Let* f *defined by* (1.1) *belong to the class* $\mathcal{H}_\Sigma(\tau, \lambda, \delta; \varphi)$ *and* $a_k = 0$ $(2 \leq k \leq n-1)$, *then*

$$|a_n| \leq \frac{B_1|\tau|}{1 + (n-1)(\lambda + n\delta)} \qquad (\ n \geq 4\ ). \tag{2.10}$$

Proof. Since $f \in \mathcal{H}_\Sigma(\tau, \lambda, \delta; \varphi)$, then we have

$$1 + \frac{1}{\tau}\left((1-\lambda)\frac{f(z)}{z} + \lambda f^{'}(z) + \delta z f^{''}(z) - 1\right) = 1 + \sum_{n=2}^{\infty}\left(\frac{1 + (n-1)(\lambda + n\delta)}{\tau}\right) a_n z^{n-1}, \tag{2.11}$$

and since the inverse map $g = f^{-1}$ represented by (2.2) also belonging to the same subclass, then

$$1 + \frac{1}{\tau}\left((1-\lambda)\frac{g(w)}{w} + \lambda g^{'}(w) + \delta z g^{''}(w) - 1\right) = 1 + \sum_{n=2}^{\infty}\left(\frac{1 + (n-1)(\lambda + n\delta)}{\tau}\right) A_n w^{n-1}. \tag{2.12}$$

Now, Since $f, g \in \mathcal{H}_\Sigma(\tau, \lambda, \delta; \varphi)$, by the definition 2.1, there exist two Schwarz functions $u(z) = \sum_{n=1}^{\infty} c_n z^n$ and $v(w) = \sum_{n=1}^{\infty} d_n w^n$ such that

$$1 + \frac{1}{\tau}\left((1-\lambda)\frac{f(z)}{z} + \lambda f^{'}(z) + \delta z f^{''}(z) - 1\right) = \varphi(u(z)), \tag{2.13}$$

$$1 + \frac{1}{\tau}\left((1-\lambda)\frac{g(w)}{w} + \lambda g^{'}(w) + \delta z g^{''}(w) - 1\right) = \varphi(v(w)), \tag{2.14}$$

such that

$$\varphi(u(z)) = 1 - \sum_{n=2}^{\infty} B_1 \mathcal{K}_{n-1}^{-1}(c_1, ..., c_{n-1}; B_1, ..., B_{n-1}) z^{n-1}, \tag{2.15}$$

$$\varphi(v(w)) = 1 - \sum_{n=2}^{\infty} B_1 \mathcal{K}_{n-1}^{-1}(d_1, ..., d_{n-1}; B_1, ..., B_{n-1}) w^{n-1}, \tag{2.16}$$

where in general $\mathcal{K}_n^p = \mathcal{K}_n^p(\rho_1, ..., \rho_n, B_1, ..., B_n)$ are defined by

$$\begin{aligned}
\mathcal{K}_n^p &= \frac{p!}{(p-n)!(n)!}\rho_1^n \frac{B_n}{B_1} + \frac{p!}{(p-n+1)!(n-2)!}\rho_1^{n-2}\rho_2 \frac{B_{n-1}}{B_1} \\
&+ \frac{p!}{(p-n+2)!(n-3)!}\rho_1^{n-3}\rho_3 \frac{B_{n-2}}{B_1} \\
&+ \frac{p!}{(p-n+3)!(n-4)!}\rho_1^{n-4}\left(\rho_4 \frac{B_{n-3}}{B_1} + \frac{p-n+3}{2}\rho_2^2 \frac{B_{n-2}}{B_1}\right) \\
&+ \frac{p!}{(p-n+4)!(n-5)!}\rho_1^{n-5}\left(\rho_5 \frac{B_{n-4}}{B_1} + (p-n+4)\rho_2\rho_3 \frac{B_{n-3}}{B_1}\right) \\
&+ \sum_{j\geq 6} \rho_1^{n-j} X_j, \qquad (2.17)
\end{aligned}$$

where X_j is a homogeneous polynomial of degree j in the variables $\rho_1, \rho_2, ..., \rho_n$.

Now, comparing the coefficients in both sides of equations (2.13) and (2.14) after substituting about $\varphi(u(z))$ and $\varphi(v(w))$ from equations (2.15) and (2.16), we have

$$\frac{1+(n-1)(\lambda+n\delta)}{\tau} a_n = -B_1 \mathcal{K}_{n-1}^{-1}(c_1, ..., c_{n-1}; B_1, ..., B_{n-1}), \qquad (2.18)$$

$$\frac{1+(n-1)(\lambda+n\delta)}{\tau} A_n = -B_1 \mathcal{K}_{n-1}^{-1}(d_1, ..., d_{n-1}; B_1, ..., B_{n-1}). \qquad (2.19)$$

Since $a_k = 0$ $(2 \leq k \leq n-1)$, then from equation (2.3) it is easy to conclude

$$A_n = -a_n. \qquad (2.20)$$

Therefore, equations (2.18) and (2.19) reduced to

$$\frac{1+(n-1)(\lambda+n\delta)}{\tau} a_n = B_1 c_{n-1}, \qquad (2.21)$$

$$-\frac{1+(n-1)(\lambda+n\delta)}{\tau} a_n = B_1 d_{n-1}. \qquad (2.22)$$

By subtracting equation (2.22) from equation (2.21) obtained

$$a_n = \frac{B_1 \tau (c_{n-1} - d_{n-1})}{2(1+(n-1)(\lambda+n\delta))}. \qquad (2.23)$$

Applying Lemma 2.1 for the coefficients c_{n-1} and d_{n-1} in equation (2.23) which reduced to the desired estimation. The proof is completed. □

By putting $\tau = 1-\alpha (0 \leq \alpha < 1)$ and $\varphi(z) = \frac{1+z}{1-z}$ $(B_1 = 2)$ in Theorem 2.1, we conclude

Corollary 2.1. [6, Theorem 2] *Let $f \in \mathcal{N}_\Sigma(\alpha, \lambda, \delta)$ and $a_k = 0$ $(2 \leq k \leq n-1)$, then*

$$|a_n| \leq \frac{2(1-\alpha)}{1+(n-1)(\lambda+n\delta)} \qquad (n \geq 4).$$

Let us put $\lambda = 1$ in Corollary 2.2, we have

Corollary 2.2. [21, Theorem 1] *Let us consider* $f \in \mathcal{N}_{\Sigma}^{(\alpha,\lambda)}$ *and* $a_k = 0 \quad (2 \leq k \leq n-1)$, *then*

$$|a_n| \leq \frac{2(1-\alpha)}{n(1+\delta(n-1))} \qquad (n \geq 4).$$

Let us put $\delta = 0$ in Corollary 2.2, we obtain

Corollary 2.3. [14, Theorem 1] *If* $f \in \mathfrak{D}(\alpha, \lambda)$ *and* $a_k = 0 \quad (2 \leq k \leq n-1)$, *then*

$$|a_n| \leq \frac{2(1-\alpha)}{1+\lambda(n-1)} \qquad (n \geq 4).$$

Theorem 2.2. *Let* $f \in \mathcal{H}_{\Sigma}(\tau, \lambda, \delta; \varphi)$ *and* $B_1 \geq |B_2|$, *then*

$$|a_2| \leq \begin{cases} \frac{B_1\sqrt{B_1}|\tau|}{\sqrt{B_1^2|\tau|(1+2\lambda+6\delta)+(B_1+B_2)(1+\lambda+2\delta)^2}} & if\ B_2 < 0, B_1 + B_2 \leq 0 \\ \\ \frac{B_1\sqrt{B_1}|\tau|}{\sqrt{B_1^2|\tau|(1+2\lambda+6\delta)+(B_1-B_2)(1+\lambda+2\delta)^2}} & if\ B_2 > 0, B_1 - B_2 \leq 0 \end{cases}, \tag{2.24}$$

$$|a_3| \leq \begin{cases} \frac{B_1|\tau|}{1+2\lambda+6\delta} & ;\ B_1 > |B_2| \\ \\ \frac{|B_2\tau|}{1+2\lambda+6\delta} & ;\ B_1 < |B_2| \end{cases}, \tag{2.25}$$

and

$$|a_3 - 2a_2^2| \leq \begin{cases} \frac{B_1|\tau|}{1+2\lambda+6\delta} & ;\ B_1 > |B_2| \\ \\ \frac{|B_2\tau|}{1+2\lambda+6\delta} & ;\ B_1 < |B_2| \end{cases}. \tag{2.26}$$

Proof. Lets us set $n = 2, n = 3$ in the equations (2.18) and (2.19), we deduce

$$\frac{1+\lambda+2\delta}{\tau} a_2 = B_1 c_1, \tag{2.27}$$

$$-\frac{1+\lambda+2\delta}{\tau} a_2 = B_1 d_1, \tag{2.28}$$

$$\frac{1+2\lambda+6\delta}{\tau} a_3 = B_1 c_2 + B_2 c_1^2, \tag{2.29}$$

and

$$\frac{1+2\lambda+6\delta}{\tau} (2a_2^2 - a_3) = B_1 d_2 + B_2 d_1^2. \tag{2.30}$$

From equations (2.27) and (2.28), we deduce

$$c_1 = -d_1, \tag{2.31}$$

and

$$a_2 = \frac{B_1 c_1 \tau}{1+\lambda+2\delta}. \tag{2.32}$$

Now, adding equation (2.29) to (2.30) obtains

$$a_2^2 = \tau \left(\frac{(B_1(c_2+d_2) + B_2(c_1^2+d_1^2))}{2(1+2\lambda+6\delta)} \right). \tag{2.33}$$

$$a_2^2 = \frac{B_1\tau}{2(1+2\lambda+6\delta)}\left[\left(c_2+\frac{B_2}{B_1}c_1^2\right)+\left(d_2+\frac{B_2}{B_1}d_1^2\right)\right]. \tag{2.34}$$

Firstly, let $B_2 < 0(\eta = \frac{B_2}{B_1} < 0,\ B_1 + B_2 \geq 0)$ and applying Lemma 2.2 with using equation (2.31), we obtain

$$|a_2|^2 \leq \frac{B_1\tau}{1+2\lambda+6\delta}\left[1-\left(\frac{B_1+B_2}{B_1}\right)|c_1|^2\right]. \tag{2.35}$$

By substituting of c_1 from equation (2.32), we conclude

$$|a_2|^2 \leq \frac{|\tau|^2 B_1^3}{B_1^2|\tau|(1+2\lambda+6\delta)+(B_1+B_2)(1+\lambda+2\delta)^2}. \tag{2.36}$$

Taking the square root of the both side of inequality (2.36), we have

$$|a_2| \leq \frac{|\tau|B_1\sqrt{B_1}}{\sqrt{B_1^2|\tau|(1+2\lambda+6\delta)+(B_1+B_2)(1+\lambda+2\delta)^2}}. \tag{2.37}$$

Second, let $B_2 > 0(\eta = \frac{B_2}{B_1} > 0,\ B_1 - B_2 \geq 0)$ and applying Lemma 2.2 with using equation (2.31), then

$$a_2^2 \leq \frac{B_1\tau}{1+2\lambda+6\delta}\left[1-\left(\frac{B_1-B_2}{B_1}\right)|c_1|^2\right]. \tag{2.38}$$

By substituting of c_1 from equation (2.32), we conclude

$$|a_2|^2 \leq \frac{|\tau|^2 B_1^3}{B_1^2|\tau|(1+2\lambda+6\delta)+(B_1-B_2)(1+\lambda+2\delta)^2}. \tag{2.39}$$

Taking the square root of the both side of inequality (2.40), we have

$$|a_2| \leq \frac{|\tau|B_1\sqrt{B_1}}{B_1^2|\tau|(1+2\lambda+6\delta)+(B_1-B_2)(1+\lambda+2\delta)^2}. \tag{2.40}$$

Combining the last inequality with inequality (2.37), we obtain the desired estimate on the coefficient $|a_2|$ which given by (2.24).

In order to deduce the estimation of $|a_3|$, subtracting equation (2.30) from (2.29) with using equation (2.32), obtains

$$a_3 = a_2^2 + \frac{B_1\tau(c_2-d_2)}{2(1+2\lambda+6\delta)}. \tag{2.41}$$

By substituting of a_2^2 from equation (2.33) into (2.41), we conclude

$$a_3 = \frac{\tau(B_1c_2+B_2c_1^2)}{1+2\lambda+6\delta}. \tag{2.42}$$

Taking the modulus of both sides of equation (2.42), we get

$$|a_3| \leq \frac{B_1|\tau|}{1+2\lambda+6\delta}\left|c_2+\frac{B_2}{B_1}c_1^2\right|. \tag{2.43}$$

By applying Lemma 2.2, let first $B_2 < 0(\eta = \frac{B_2}{B_1} < 0)$, then

$$|a_3| \leq \frac{B_1|\tau|}{1+2\lambda+6\delta}\left[1-\frac{B_1-B_2}{B_1}|c_1|^2\right]. \tag{2.44}$$

If $B_1 - B_2 > 0$, then we must put $|c_1|$ by its least value $|c_1| = 0$. Thus

$$|a_3| \leq \frac{B_1|\tau|}{1+2\lambda+6\delta}. \tag{2.45}$$

If $B_1 - B_2 < 0$, then we must put $|c_1|$ by its maximum value $|c_1| = 1$ (using Lemma 2.2). Thus

$$|a_3| \leq \frac{B_2|\tau|}{1+2\lambda+6\delta}. \tag{2.46}$$

Second, let us put $B_2 > 0 (\eta = \frac{B_2}{B_1} > 0)$, then

$$|a_3| \leq \frac{B_1|\tau|}{1+2\lambda+6\delta}\left[1 - \frac{B_1+B_2}{B_1}|c_1|^2\right]. \tag{2.47}$$

If $B_1 + B_2 > 0$, then we must put $|c_1|$ by its least value $|c_1| = 0$. Thus

$$|a_3| \leq \frac{B_1|\tau|}{1+2\lambda+6\delta}. \tag{2.48}$$

If $B_1 + B_2 < 0$, then we must put $|c_1|$ by its maximum value $|c_1| = 1$ (using Lemma 2.1). Thus

$$|a_3| \leq \frac{-B_2|\tau|}{1+2\lambda+6\delta}. \tag{2.49}$$

By comparing the estimates of $|a_3|$ in relations from (2.45) to (2.48) which obtain the desired estimate given by (2.25). Finally, using equation (2.30), gives

$$a_3 - 2a_2^2 = \frac{-\tau(B_1 d_2 + B_2 d_1^2)}{1+2\lambda+6\delta}. \tag{2.50}$$

Using the same technique in proving the estimate of $|a_3|$, we get the desired estimate given by (2.26), then we prefer to omit it. □

In case of $\lambda = 1$, Theorem 2.2 becomes

Corollary 2.4. [22, Theorem 1] *Let $f \in \Sigma(\tau, \delta, \varphi)$, then*

$$|a_2| \leq \begin{cases} \frac{B_1\sqrt{B_1}|\tau|}{\sqrt{3B_1^2|\tau|(1+2\delta)+4(B_1+B_2)(1+\delta)^2}} & B_2 < 0 \text{ and } B_1 + B_2 \geq 0 \\ \\ \frac{B_1\sqrt{B_1}|\tau|}{\sqrt{3B_1^2|\tau|(1+2\delta)+4(B_1-B_2)(1+\delta)^2}} & B_2 > 0 \text{ and } B_1 - B_2 \geq 0 \end{cases},$$

$$|a_3| \leq \begin{cases} \frac{B_1|\tau|}{3(1+2\delta)} & B_1 > |B_2| \\ \\ \frac{|B_2\tau|}{3(1+2\delta)} & B_1 < |B_2| \end{cases}.$$

Let us put $\varphi(z) = \left(\frac{1+z}{1-z}\right)^\alpha$, $B_1 = 2\alpha$ and $B_2 = 2\alpha^2$, and $\tau = 1$ in Corollary 2.4 we have

Corollary 2.5. [11, Theorem 2.2] *Let $f \in \mathcal{H}_\Sigma(\alpha, \delta)$, then*

$$|a_2| \leq \frac{2\alpha}{\sqrt{2(2+\alpha) + 4\delta(\alpha+\delta-\alpha\delta+2)}},$$

$$|a_3| \leq \frac{2\alpha}{3(1+2\delta)}.$$

By putting $\tau = 1-\gamma$ and $\varphi(z) = \frac{1+z}{1-z}$, $B_1 = B_2 = 2$, in Corollary 2.4, we obtain

Corollary 2.6. [11, Theorem 3.2] *Let* $f \in \mathcal{H}_\Sigma(\gamma, \delta)$*, then*

$$|a_2| \leq \sqrt{\frac{2(1-\gamma)}{3(1+2\delta)}}, \quad |a_3| \leq \frac{2(1-\gamma)}{3(1+2\delta)}.$$

In case of $\tau = 1$, $\delta = 0$ and $\varphi(z) = \left(\frac{1+z}{1-z}\right)^\alpha$, $B_1 = 2\alpha, B_2 = 2\alpha^2$, in Theorem 2.2, we have

Corollary 2.7. [10, Theorem 2.2] *Let* $f \in \mathcal{B}_\Sigma(\alpha, \lambda)$*, then*

$$|a_2| \leq \frac{2\alpha}{\sqrt{(1+\lambda)^2 + \alpha(1+2\lambda-\lambda^2)}}, \quad |a_3| \leq \frac{2\alpha}{1+2\lambda}.$$

Let us put $\tau = 1-\gamma$ and $\varphi(z) = \frac{1+z}{1-z}$, $B_1 = B_2 = 2$, in Theorem 2.2, we obtain

Corollary 2.8. [6, Theorem 5] *Let* $0 \leq \alpha < 1$ *and* $f \in \mathcal{N}_\Sigma(\gamma, \lambda, \delta)$*, then*

$$|a_2| \leq \sqrt{\frac{2(1-\gamma)}{1+2\lambda+6\delta}},$$

$$|a_3| \leq \frac{2(1-\gamma)}{1+2\lambda+6\delta},$$

and

$$|a_3 - 2a_2^2| \leq \frac{2(1-\gamma)}{1+2\lambda+6\delta}.$$

By putting $\delta = 0$ in Corollary 2.8, gets

Corollary 2.9. [10, Theorem 3.2] *If* f *belong to* $\mathcal{B}_\Sigma(\gamma, \lambda)$ *and* $0 \leq \gamma < 1$*, then*

$$|a_2| \leq \sqrt{\frac{2(1-\gamma)}{1+2\lambda}},$$

$$|a_3| \leq \frac{2(1-\gamma)}{1+2\lambda},$$

Remark 2.2. Some results investigated in Corollaries from 2.4 to 2.9 represented an improvement of the estimate of $|a_3|$ of the earlier corresponding results.

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