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IMPROVED CAUCHY-SCHWARZ INEQUALITY AND ITS APPLICATIONS

SLOBODAN FILIPOVSKI[1]

Abstract. In this paper we present an improvement of the well-known Cauchy-Schwarz inequality in $\mathbb{R}^n$. Based on this improvement, we improve the inequality between quadratic and arithmetic mean of n positive real numbers and we give a new refinement of the triangle inequality in $\mathbb{R}^n$.

1. Introduction

The Cauchy-Schwarz inequality is one of the most famous inequalities in mathematics. Let $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$ be two sequences in Euclidean space $\mathbb{R}^n$ with the standard inner product. The remarkable Cauchy-Schwarz inequality states

$$\sqrt{\sum_{i=1}^{n} a_i^2} \cdot \sqrt{\sum_{i=1}^{n} b_i^2} \geq \sum_{i=1}^{n} a_i b_i$$

with equality if and only if a and b are proportional, see [3].
Under additional conditions the inequalities can be improved. Notable refinement of the Cauchy-Schwarz inequality is given in [8]. Ostrowski showed that if $a = (a_1, \ldots, a_n), b = (b_1, \ldots, b_n)$ and $c = (c_1, \ldots, c_n)$ are n-tuples of real numbers such that a and b are not proportional and $\sum_{i=1}^{n} a_i c_i = 0$ and $\sum_{i=1}^{n} b_i c_i = 1$, then

$$\sum_{i=1}^{n} a_i^2 \cdot \sum_{i=1}^{n} b_i^2 \geq \left(\sum_{i=1}^{n} a_i b_i\right)^2 + \frac{\sum_{i=1}^{n} a_i^2}{\sum_{i=1}^{n} c_i^2}.$$

Another known refinement of the Cauchy-Schwarz inequality is established in [1]. Alzer proved, if $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$ are two sequences of real numbers and $0 = a_0 < a_1 \leq \frac{a_2}{2} \leq \ldots \leq \frac{a_n}{n}$ and $0 < b_n \leq b_{n-1} \leq \ldots \leq b_1$, then

$$\sum_{i=1}^{n} b_i \sum_{i=1}^{n} \left(a_i^2 - \frac{a_i a_{i-1}}{4}\right) b_i \geq \left(\sum_{i=1}^{n} a_i b_i\right)^2.$$

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Significant improvements of the Cauchy-Schwarz inequality in complex inner product spaces are given by Dragomir in [4]. In [9], Walker gives refinement of this inequality in probability case.
Relied on a relatively simple idea, Lemma 2.1, in this paper we give a new improvement of Cauchy-Schwarz inequality, Theorem 2.1. We see the main strength of our result in providing new bound without any additional conditions on the parameters a_i and b_i. As a consequence of the improved Cauchy-Schwarz inequality we obtain two new improvements; on the inequality between Quadratic and Arithmetic mean, Theorem 3.1, and on the triangle inequality in $\mathbb{R}^n$, Theorem 4.3.

2. Improved Cauchy-Schwarz inequality

It is well known that for any two strictly positive real numbers a and b it occurs $\sqrt{\frac{a}{b}} + \sqrt{\frac{b}{a}} \geq 2$. The main result in this paper is based on the following lemma, which presents an improvement of the above inequality.

Lemma 2.1. *If x and y are strictly positive real numbers, then*

$$\sqrt{\frac{x}{y}} + \sqrt{\frac{y}{x}} \geq 2 + \frac{(x-y)^2}{2(x^2+y^2)}.$$

Proof. We prove the following equivalent inequality

$$\sqrt{\frac{x}{y}} + \sqrt{\frac{y}{x}} \geq 2 + \frac{\left(\frac{x}{y}\right)^2 - 2\left(\frac{x}{y}\right) + 1}{2\left(\left(\frac{x}{y}\right)^2 + 1\right)}.$$

Let $t^2 = \frac{x}{y}, t > 0$. The required inequality is equivalent to the inequalities

$$t + \frac{1}{t} \geq 2 + \frac{t^4 - 2t^2 + 1}{2(t^4+1)} \Leftrightarrow 2t^6 - 5t^5 + 2t^4 + 2t^3 + 2t^2 - 5t + 2 \geq 0.$$

Now we easily show $2t^6 - 5t^5 + 2t^4 + 2t^3 + 2t^2 - 5t + 2 = (t-1)^4(2t^2 + 3t + 2) \geq 0$. □

The main result in this paper is the following theorem.

Theorem 2.1. *Let $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$ be two sequences of positive real numbers such that $a_i^2 + b_i^2 \neq 0$ for each $i = 1, \ldots, n$. Let $A = \sqrt{\sum_{i=1}^n a_i^2}$ and $B = \sqrt{\sum_{i=1}^n b_i^2}$ and let $A, B \neq 0$. Then the following inequality holds*

$$\sqrt{\sum_{i=1}^n a_i^2} \cdot \sqrt{\sum_{i=1}^n b_i^2} \geq \sum_{i=1}^n a_i b_i + \frac{1}{4} \cdot \sum_{i=1}^n \frac{(a_i^2 B^2 - b_i^2 A^2)^2}{a_i^4 B^4 + b_i^4 A^4} a_i b_i \tag{2.1}$$

The equality holds if and only if a and b are proportional.

Proof. Setting $x = \frac{a_i^2}{A^2}$ and $y = \frac{b_i^2}{B^2}$ in Lemma 2.1 we get

$$\frac{a_i^2}{A^2} + \frac{b_i^2}{B^2} \geq \left(2 + \frac{(a_i^2 B^2 - b_i^2 A^2)^2}{2(a_i^4 B^4 + b_i^4 A^4)}\right) \frac{a_i b_i}{AB}. \tag{2.2}$$

If we fix $i = 1, \ldots, n$ in the inequality (2.2), and if we sum up the obtained n inequalities, we get

$$\sum_{i=1}^{n}\left(\frac{a_i^2}{A^2}+\frac{b_i^2}{B^2}\right) \geq \frac{1}{AB}\left(2\sum_{i=1}^{n} a_ib_i + \frac{1}{2}\sum_{i=1}^{n}\frac{(a_i^2B^2-b_i^2A^2)^2}{a_i^4B^4+b_i^4A^4}a_ib_i\right). \tag{2.3}$$

Now, the inequality in (2.1) follows directly from (2.3) using $\sum_{i=1}^{n}\frac{a_i^2}{A^2}+\sum_{i=1}^{n}\frac{b_i^2}{B^2}=2$. If a and b are proportional, then $\frac{a_i}{b_i}=\frac{A}{B}$, i.e. $a_i^2B^2-b_i^2A^2=0$ for each $i = 1, 2, \ldots, n$. In this case holds the equality $\sqrt{\sum_{i=1}^{n} a_i^2}\cdot\sqrt{\sum_{i=1}^{n} b_i^2}=\sum_{i=1}^{n} a_ib_i$. □

3. Improved inequality between Quadratic and Arithmetic mean

Between quadratic and arithmetic mean of n positive real numbers $a_1, \ldots, a_n$ the following inequality holds:

$$\sqrt{\frac{a_1^2+\ldots+a_n^2}{n}} \geq \frac{a_1+\ldots+a_n}{n}.$$

The well-known QM-AM inequality is equivalent to the inequality

$$n(a_1^2+\ldots+a_n^2) \geq (a_1+\ldots+a_n)^2 \tag{3.1}$$

which follows easy if we apply Cauchy-Schwarz inequality to the sequences $a = (a_1, \ldots, a_n)$ and $b = (1, \ldots, 1)$. Using Theorem 2.1, we are in a position to improve the inequality in (3.1).

Lemma 3.1. *Let $a_1, \ldots, a_n$ be positive real numbers such that $a_1^2+\ldots+a_n^2 \neq 0$. Then the following inequality holds*

$$\sqrt{n}\cdot\sqrt{a_1^2+\ldots+a_n^2} \geq a_1+\ldots+a_n+\frac{1}{4}\cdot\sum_{i=1}^{n}\frac{(na_i^2-(a_1^2+\ldots+a_n^2))^2}{n^2a_i^4+(a_1^2+\ldots+a_n^2)^2}a_i. \tag{3.2}$$

Proof. It follows directly from (2.1), setting $b_1 = \ldots = b_n = 1$ and $B = \sqrt{n}$. □

If we divide the inequality in (3.2) by n we arrive to a new refinement of the famous quadratic-arithmetic mean as follows.

Theorem 3.1. *If $a_1, \ldots, a_n$ are n positive real numbers such that $a_1^2+\ldots+a_n^2 \neq 0$, then*

$$\sqrt{\frac{a_1^2+\ldots+a_n^2}{n}} \geq \frac{a_1+\ldots+a_n}{n}+\frac{1}{4n}\cdot\sum_{i=1}^{n}\frac{(na_i^2-(a_1^2+\ldots+a_n^2))^2}{n^2a_i^4+(a_1^2+\ldots+a_n^2)^2}a_i.$$

4. A new refinement of the triangle inequality

For any two vectors x and y in the normed linear space $(X, \|\cdot\|)$ over the real or complex numbers occurs the triangle inequality $\|x+y\| \leq \|x\|+\|y\|$. Among many known refinements of the triangle inequality let us mention two of them. In [5,6] Maligranda proved a refinement of the triangle inequality as follows:

Theorem 4.1. *For nonzero vectors x and y in a normed space $(X, \|\cdot\|)$, it is true that*

$$\left(2 - \left\|\frac{x}{\|x\|} + \frac{y}{\|y\|}\right\|\right) \min\{\|x\|, \|y\|\} \leq \|x\| + \|y\| - \|x + y\| \leq$$

$$\leq \left(2 - \left\|\frac{x}{\|x\|} + \frac{y}{\|y\|}\right\|\right) \max\{\|x\|, \|y\|\}.$$

An improvement of the inequality due to Maligranda is done by Minculete and Păltănea in [7]. Using integrals and the Tapia semi-product they proved the following result:

Theorem 4.2. *Let $(X, \langle\cdot,\cdot\rangle)$ be an inner product space, with norm $\|\cdot\|$. For nonzero elements $x, y \in X$*

$$\|x\| + \|y\| - \|x + y\| \leq \left(1 - \frac{1}{2}\|v(x, y)\|\right)(\|x\| + \|y\|),$$

where,

$$\|v(x, y)\| = \left\|\frac{x}{\|x\|} + \frac{y}{\|y\|}\right\| = \sqrt{2\left(1 + \frac{\langle x, y\rangle}{\|x\| \cdot \|y\|}\right)}.$$

In this paper, based on the improved Cauchy-Schwarz inequality, we give a new refinement of the triangle inequality in the Euclidean space $\mathbb{R}^n$ with the standard inner product.

Theorem 4.3. *Let $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ be two vectors in $\mathbb{R}^n$ such that $x, y \neq 0$ and $x_i^2 + y_i^2 \neq 0$ for each $i = 1, \ldots, n$. Then*

$$\|x + y\|^2 + \frac{1}{2} \cdot \sum_{i=1}^{n} \frac{(x_i^2\|y\|^2 - y_i^2\|x\|^2)^2}{x_i^4\|y\|^4 + y_i^4\|x\|^4} x_i y_i \leq (\|x\| + \|y\|)^2.$$

Proof. Since $x + y = (x_1 + y_1, \ldots, x_n + y_n)$, $\|x\| = \sqrt{x_1^2 + \ldots + x_n^2}$ and $\|y\| = \sqrt{y_1^2 + \ldots + y_n^2}$ we have

$$\|x + y\|^2 + \frac{1}{2} \cdot \sum_{i=1}^{n} \frac{(x_i^2\|y\|^2 - y_i^2\|x\|^2)^2}{x_i^4\|y\|^4 + y_i^4\|x\|^4} x_i y_i = \sum_{i=1}^{n} x_i^2 + \sum_{i=1}^{n} y_i^2 +$$

$$+2\left(\sum_{i=1}^{n} x_i y_i + \frac{1}{4} \cdot \sum_{i=1}^{n} \frac{(x_i^2\|y\|^2 - y_i^2\|x\|^2)^2}{x_i^4\|y\|^4 + y_i^4\|x\|^4} x_i y_i\right) \leq (\|x\| + \|y\|)^2.$$

□

Remark 4.1. If $\|x\| = \|y\|$ and $x_i^2 + y_i^2 \neq 0$ for each $i = 1, \ldots, n$, then the improved triangle inequality becomes

$$\|x + y\|^2 + \frac{1}{2} \cdot \sum_{i=1}^{n} \frac{(x_i^2 - y_i^2)^2}{x_i^4 + y_i^4} x_i y_i \leq 4\|x\|^2.$$

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[1]Department of Mathematics,
University of Primorska,
Kettejeva 1, Koper
E-mail address: slobodan.filipovski@famnit.upr.si