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INEQUALITIES ARISING FROM THE MONOTONICITY AND CONVEXITY

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In memory of our friend Hatice Yaldız

ABSTRACT. We study the possibilities of creating inequalities with monotonic and convex functions. This research makes the most of the convex combination center and bounded set barycenter. Such approach gives rise to multiple inequalities in which the middle members are variable.

1. INTRODUCTION

If $a, b \in \mathbb{R}$ are points such that $a < b$, then the closed interval with endpoints a and b can be introduced as the set

$$[a, b] = \{\alpha a + \beta b : \alpha, \beta \in [0, 1],\ \alpha + \beta = 1\}. \tag{1.1}$$

Each point $x \in [a, b]$ is represented by the unique binomial convex combination $\alpha_x a + \beta_x b$ because the difference $a - b$ is not zero. The convex combination

$$x = \frac{b - x}{b - a} a + \frac{x - a}{b - a} b$$

shows that $\alpha_x = (b - x)/(b - a)$ and $\beta_x = (x - a)/(b - a)$. Formula (1.1) also applies in the case $a = b$ where $[a, a] = \{a\}$. If we use formula (1.1) with $\alpha, \beta \in (0, 1)$, then we have the open interval (a, b).

The convex hull of a set $X \subseteq \mathbb{R}$ is the set $\mathrm{conv}X$ containing all convex combinations of points from X.

Let $\sum_{i=1}^{n} \lambda_i x_i$ be a convex combination of points $x_i \in [a, b]$. Its center c can be formally defined by the equation $\sum_{i=1}^{n} \lambda_i (x_i - c) = 0$. Thus $c = \sum_{i=1}^{n} \lambda_i x_i$ and it belongs to the

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convex hull of $\{x_1, \ldots, x_n\}$. A convex function $f : [a, b] \to \mathbb{R}$ satisfies the discrete form of Jensen's inequality (see [4])

$$f\left(\sum_{i=1}^{n} \lambda_i x_i\right) \leq \sum_{i=1}^{n} \lambda_i f(x_i). \tag{1.2}$$

Let $X \subseteq [a, b]$ be a set of the length $|X| > 0$. Its barycenter $\bar{c}$ can be defined by the integral equation $\int_X (x - \bar{c})\, dx = 0$. Thus $\bar{c} = \int_X x\, dx / |X|$ and it belongs to the convex hull of X. A convex function $f : [a, b] \to \mathbb{R}$ satisfies the simple integral form of Jensen's inequality (see [5])

$$f\left(\frac{\int_X x\, dx}{|X|}\right) \leq \frac{\int_X f(x)\, dx}{|X|}. \tag{1.3}$$

If we use an affine function f, then we have the equalities in formula (1.2) and formula (1.3).

2. Inequalities on the bounded closed interval

The consequent four lemmas employ the common centers and barycenters.

Lemma 2.1. *Let $\sum_{i=1}^{n} \lambda_i x_i$ be a convex combination of points $x_i \in [a, b]$, and let $\alpha a + \beta b$ be the convex combination of the endpoints a and b such that*

$$\sum_{i=1}^{n} \lambda_i x_i = \alpha a + \beta b.$$

Then each convex function $f : [a, b] \to \mathbb{R}$ satisfies the double inequality

$$f(\alpha a + \beta b) \leq \sum_{i=1}^{n} \lambda_i f(x_i) \leq \alpha f(a) + \beta f(b). \tag{2.1}$$

Lemma 2.2. *Let $X \subseteq [a, b]$ be a set of positive length, and let $\alpha a + \beta b$ be the convex combination of the endpoints a and b such that*

$$\frac{\int_X x\, dx}{|X|} = \alpha a + \beta b.$$

Then each convex function $f : [a, b] \to \mathbb{R}$ satisfies the double inequality

$$f(\alpha a + \beta b) \leq \frac{\int_X f(x)\, dx}{|X|} \leq \alpha f(a) + \beta f(b). \tag{2.2}$$

Lemma 2.3. *Let $\sum_{i=1}^{n} \lambda_i x_i$ and $\sum_{j=1}^{m} \kappa_j y_j$ be convex combinations of points $x_i, y_j \in [a, b]$ such that no y_j belongs to the interior of* $\operatorname{conv}\{x_1, \ldots, x_n\}$ *and that*

$$\sum_{i=1}^{n} \lambda_i x_i = \sum_{j=1}^{m} \kappa_j y_j,$$

and let λ and κ be nonnegative numbers such that $\lambda + \kappa = 1$.

Then each convex function $f : [a, b] \to \mathbb{R}$ satisfies the double inequality

$$\sum_{i=1}^{n} \lambda_i f(x_i) \leq \lambda \sum_{i=1}^{n} \lambda_i f(x_i) + \kappa \sum_{j=1}^{m} \kappa_j f(y_j) \leq \sum_{j=1}^{m} \kappa_j f(y_j). \tag{2.3}$$

The presentation of the middle member in formula (2.3) is not substantial. Its full meaning comes to expression in the integral form.

Lemma 2.4. *Let $X, Y \subseteq [a, b]$ be sets of positive lengths such that Y does not intersect the interior of* $\operatorname{conv} X$ *and that*

$$\frac{\int_X x\,dx}{|X|} = \frac{\int_Y y\,dy}{|Y|}.$$

Then each convex function $f : [a, b] \to \mathbb{R}$ satisfies the double inequality

$$\frac{\int_X f(x)\,dx}{|X|} \leq \frac{\int_{X\cup Y} f(z)\,dz}{|X \cup Y|} \leq \frac{\int_Y f(y)\,dy}{|Y|}. \tag{2.4}$$

The above four Jensen type inequalities show that the convex function values taken in the form of convex combinations or integral arithmetic means grow from the center, across the middle, to the ends.

The above inequalities can be proved by using the secant line $y = h(x)$ over the convex hull of $\{x_1, \ldots, x_n\}$ or X, and applying the convexity of f and affinity of h through their discrete and integral forms. The special cases for the discrete inequalities in formula (2.1) and formula (2.3) must be considered particularly.

The inequality in formula (2.3) can be extended by inserting into the inequality $f(\alpha a + \beta b) \leq \alpha f(a) + \beta f(b)$ if $\alpha a + \beta b = \sum_{i=1}^n \lambda_i x_i = \sum_{j=1}^m \kappa_j y_j$. The same is true for the inequality in formula (2.4) if $\alpha a + \beta b = \int_X x\,dx/|X| = \int_Y y\,dy/|Y|$.

The discrete Jensen type inequality in formula (2.3) and the integral Jensen type inequality in formula (2.4) were generally discussed in [13]. New generalizations and refinements of the Jensen inequality were considered in [6], [7] and [8].

3. Main results

Initially, we present a simple construction of a nondecreasing and convex function which can be employed to create inequalities.

Theorem 3.1. *Let $f : [a, b] \to \mathbb{R}$ be a convex function, and let $c = \alpha a + \beta b$ be a convex combination of the endpoints a and b.*

Then the function $f_c : [0, 1] \to \mathbb{R}$ defined by

$$f_c(t) = \alpha f\big(ta + (1 - t)c\big) + \beta f\big((1 - t)c + tb\big) \tag{3.1}$$

is nondecreasing and convex.

Proof. The representation of f_c: By pointing out the convex combinations originating from the point c and determined as

$$a_c(t) = ta + (1 - t)c \ \text{ and } \ b_c(t) = (1 - t)c + tb, \tag{3.2}$$

we can set up the representation

$$f_c(t) = \alpha f(a_c(t)) + \beta f(b_c(t)). \tag{3.3}$$

The c-originating combinations in formula (3.2) can be regarded as the restricted affine functions $a_c : [0,1] \to [a,c]$ and $b_c : [0,1] \to [c,b]$.

The monotonicity of f_c: The proof of this part is based on Lemma 2.1. We take a pair of arguments $t_1, t_2 \in [0,1]$ such that $t_1 < t_2$. Then it follows that

$$a_c(t_2) \leq a_c(t_1) \leq c \leq b_c(t_1) \leq b_c(t_2).$$

So we have the inclusion $a_c(t_1), b_c(t_1) \in [a_c(t_2), b_c(t_2)]$. We also have the equality

$$\alpha a_c(t_1) + \beta b_c(t_1) = c = \alpha a_c(t_2) + \beta b_c(t_2).$$

By applying the right-hand side (containing the second and third members) of the double inequality in formula (2.1) to the above inclusion and equality, we obtain the inequality

$$\alpha f(a_c(t_1)) + \beta f(b_c(t_1)) \leq \alpha f(a_c(t_2)) + \beta f(b_c(t_2))$$

which determines the relation $f_c(t_1) \leq f_c(t_2)$. Thus f_c is nondecreasing.

The convexity of f_c: Since the compositions $f \circ a_c$ and $f \circ b_c$ are convex, their convex combination $f_c = \alpha(f \circ a_c) + \beta(f \circ b_c)$ is certainly convex. □

If $c = a$ or $c = b$, then $f_c(t) = f(c)$ for every $t \in [0,1]$, and so the constant $f(c)$ represents the function f_c. The same is true if the function f is affine.

The combinations $a_c(t) = ta + (1-t)c$ and $b_c(t) = (1-t)c + tb$ tend to the center c if t tends to 0, and tend to the ends a and b if t tends to 1. The combinations $a_c^\star(t) = (1-t)a + tc$ and $b_c^\star(t) = tc + (1-t)b$ behave the opposite.

Remark 3.1. Let $f : [a,b] \to \mathbb{R}$ be a convex function, and let $c = \alpha a + \beta b$ be a convex combination of the endpoints a and b.

Then the function $f_c^\star : [0,1] \to \mathbb{R}$ defined by

$$f_c^\star(t) = \alpha f\big((1-t)a + tc\big) + \beta f\big(tc + (1-t)b\big)$$

is nonincreasing and convex.

The effects of the function f_c will be presented through the inequalities exposed in corollaries and theorems that follow.

Corollary 3.1. *Let $c = \alpha a + \beta b$ be a convex combination of the endpoints a and b, and let $t \in [0,1]$ be a number.*

Then each convex function $f : [a,b] \to \mathbb{R}$ satisfies the double inequality

$$f(\alpha a + \beta b) \;\leq\; \alpha f(ta + (1-t)c) + \beta f((1-t)c + tb) \;\leq\; \alpha f(a) + \beta f(b). \tag{3.4}$$

Proof. The nondecreasing function f_c provides a simple proof. Since

$$f_c(0) = \alpha f(c) + \beta f(c) = f(c) = f(\alpha a + \beta b)$$

and

$$f_c(1) = \alpha f(a) + \beta f(b),$$

the relations

$$f_c(0) \leq f_c(t) \leq f_c(1)$$

affirm the inequality in formula (3.4). □

The inequality in formula (3.4) contains the variable number t. It allows the integration over the interval $[0,1]$ by the variable t.

Corollary 3.2. *Let $c = \alpha a + \beta b$ be a convex combination of the endpoints a and b with positive coefficients α and β.*

Then each convex function $f : [a,b] \to \mathbb{R}$ satisfies the double inequality

$$f(\alpha a + \beta b) \leq \alpha \frac{\int_a^c f(x)\,dx}{c-a} + \beta \frac{\int_c^b f(x)\,dx}{b-c} \leq \alpha f(a) + \beta f(b). \tag{3.5}$$

By using $c = (a+b)/2$ in formula (3.5), or using $X = [a,b]$ in formula (2.2), we get the Hermite-Hadamard inequality (see [3] and [2])

$$f\left(\frac{a+b}{2}\right) \leq \frac{\int_a^b f(x)\,dx}{b-a} \leq \frac{f(a)+f(b)}{2}. \tag{3.6}$$

As a reciprocity, this useful inequality gives an opportunity to refine the inequality in formula (3.5). The Jensen inequality joins too.

Corollary 3.3. *Let $c = \alpha a + \beta b$ be a convex combination of the endpoints a and b with positive coefficients α and β.*

Then each convex function $f : [a,b] \to \mathbb{R}$ satisfies the multiple inequality

$$\begin{aligned} f(\alpha a+\beta b) \;&\leq\; \alpha f\left(\frac{a+c}{2}\right) + \beta f\left(\frac{c+b}{2}\right) \;\leq\; \alpha \frac{\int_a^c f(x)dx}{c-a} + \beta \frac{\int_c^b f(x)dx}{b-c} \\ &\leq\; \alpha\, \frac{f(a)+f(c)}{2} + \beta\, \frac{f(c)+f(b)}{2} \;\leq\; \alpha f(a) + \beta f(b). \end{aligned} \tag{3.7}$$

Proof. The inequality of the first and second members follows from the Jensen inequality applied to the convex combinations equality

$$\alpha a + \beta b = c = \alpha \frac{a+c}{2} + \beta \frac{c+b}{2}.$$

The inequality of the second, third and fourth members follows from the Hermite–Hadamard inequality applied to the function f_c as

$$f_c\left(\frac{0+1}{2}\right) \leq \int_0^1 f_c(t)\,dt \leq \frac{f_c(0)+f_c(1)}{2}.$$

The inequality of the fourth and fifth members follows from the convexity inequality $f(c) \leq \alpha f(a) + \beta f(b)$. □

To generalize Lemma 2.1, we use the c-originating combinations as points.

Theorem 3.2. *Let $c = \sum_{i=1}^n \lambda_i x_i$ be a convex combination of points $x_i \in [a,b]$, let $c = \alpha a + \beta b$ be the convex combination of the endpoints a and b, and let $t \in [0,1]$ be a number.*

Then each convex function $f:[a,b]\to\mathbb{R}$ satisfies the multiple inequality

$$\begin{aligned} f(\alpha a+\beta b) &\le \sum_{i=1}^{n}\lambda_i f\big(tx_i+(1-t)c\big) \le \alpha f\big(ta+(1-t)c\big)+\beta f\big((1-t)c+tb\big) \\ &\le \alpha f(a)+\beta f(b). \end{aligned} \tag{3.8}$$

Proof. The inequality of the last two members is included in formula (3.4).

To provide the inequality of the first three members, we use the c-originating convex combinations

$$a_c(t)=ta+(1-t)c,\;\; x_{ic}(t)=tx_i+(1-t)c,\;\; b_c(t)=(1-t)c+tb.$$

Each point x_i can be represented by the convex combination $x_i=\alpha_i a+\beta_i b$. It further gives the similar convex combination

$$x_{ic}(t)=\alpha_i a_c(t)+\beta_i b_c(t).$$

This implies the inclusion $x_{ic}(t)\in[a_c(t),b_c(t)]$ for $i=1,\dots,n$. Besides, there follows the equality

$$\sum_{i=1}^{n}\lambda_i x_{ic}(t)=c=\alpha a_c(t)+\beta b_c(t).$$

We can apply the double inequality in formula (2.1) to the above inclusion and equality, and obtain the inequality

$$f\big(\alpha a_c(t)+\beta b_c(t)\big)\le\sum_{i=1}^{n}\lambda_i f\big(x_{ic}(t)\big)\le\alpha f\big(a_c(t)\big)+\beta f\big(b_c(t)\big)$$

which provides the inequality of the first three members. □

The integration of formula (3.8) assumes and yields the following.

Corollary 3.4. *Let $c=\sum_{i=1}^{n}\lambda_i x_i$ be a convex combination of points $x_i\in[a,b]$ such that no x_i coincides with c, and let $c=\alpha a+\beta b$ be the convex combination of the endpoints a and b.*

Then each convex function $f:[a,b]\to\mathbb{R}$ satisfies the multiple inequality

$$\begin{aligned} f(\alpha a+\beta b) &\le \sum_{i=1}^{n}\lambda_i\frac{\int_c^{x_i}f(x)dx}{x_i-c} \le \alpha\frac{\int_a^{c}f(x)dx}{c-a}+\beta\frac{\int_c^{b}f(x)dx}{b-c} \\ &\le \alpha f(a)+\beta f(b). \end{aligned} \tag{3.9}$$

In the next generalization of Lemma 2.3, we only observe the end-members of the inequality in formula (2.3).

Theorem 3.3. *Let $c=\sum_{i=1}^{n}\lambda_i x_i$ and $c=\sum_{j=1}^{m}\kappa_j y_j$ be convex combinations of points $x_i,y_j\in[a,b]$ such that no y_j belongs to the interior of the convex hull of $\{x_1,\dots,x_n\}$, let $c=\alpha a+\beta b$ be the convex combination of the endpoints a and b, and let $t\in[0,1]$ be a number.*

Then each convex function $f:[a,b]\to\mathbb{R}$ satisfies the multiple inequality

$$\begin{aligned} f(\alpha a+\beta b) &\le \sum_{i=1}^{n}\lambda_i f\big(tx_i+(1-t)c\big) \le \sum_{j=1}^{m}\kappa_j f\big(ty_j+(1-t)c\big) \\ &\le \alpha f(a)+\beta f(b). \end{aligned} \tag{3.10}$$

Proof. The inequalities of the first two members and last two members are covered by formula (3.8).

To prove the inequality of the middle members, we emphasize the numbers

$$a_0=\min\{x_1,\dots,x_n\} \quad\text{and}\quad b_0=\max\{x_1,\dots,x_n\},$$

combinations $x_{ic}(t)=tx_i+(1-t)c$ and $y_{jc}(t)=ty_j+(1-t)c$, and equality

$$\sum_{i=1}^{n}\lambda_i x_{ic}(t)=c=\sum_{j=1}^{m}\kappa_j y_{jc}(t). \tag{3.11}$$

If $a_0=b_0$, then $x_1=\ldots=x_n=c$. There is no restriction on the points y_j. To represent the inequality of the middle members, we engage Jensen's inequality within the relations

$$\sum_{i=1}^{n}\lambda_i f(x_{ic}(t))=f(c)\le\sum_{j=1}^{m}\kappa_j f(y_{jc}(t)). \tag{3.12}$$

If $a_0<b_0$, then we engage the c-originating convex combinations

$$a_{0c}(t)=ta_0+(1-t)c \quad\text{and}\quad b_{0c}(t)=(1-t)c+tb_0.$$

Each point y_j can be represented by the affine combination $y_j=\alpha_j a_0+\beta_j b_0$ including $\alpha_j\le 0$ or $\beta_j\le 0$ because $y_j\notin(a_0,b_0)$. It follows that

$$y_{jc}(t)=\alpha_j a_{0c}(t)+\beta_j b_{0c}(t),$$

and so $y_{jc}(t)\notin(a_{0c}(t),b_{0c}(t))$. Further, each $x_{ic}(t)\in[a_{0c}(t),b_{0c}(t)]\subseteq[a_c(t),b_c(t)]$ and each $y_{jc}(t)\in[a_c(t),b_c(t)]$ by the proof of Theorem 3.2. We have the inclusions

$$x_{ic}(t)\in[a_{0c}(t),b_{0c}(t)] \quad\text{and}\quad y_{jc}(t)\in[a_c(t),b_c(t)]\setminus(a_{0c}(t),b_{0c}(t)).$$

By applying the inequality of the end-members in formula (2.3) to the above inclusions and the equality in formula (3.11), we obtain the inequality of the end-members in formula (3.12). □

The integration of the inequality in formula (3.10) runs as follows.

Corollary 3.5. *Let $c=\sum_{i=1}^{n}\lambda_i x_i$ and $c=\sum_{j=1}^{m}\kappa_j y_j$ be convex combinations of points $x_i,y_j\in[a,b]$ such that no y_j belongs to the interior of the convex hull of $\{x_1,\dots,x_n\}$ and that no x_i or y_j coincides with c, and let $c=\alpha a+\beta b$ be the convex combination of the endpoints a and b.*

Then each convex function $f:[a,b]\to\mathbb{R}$ satisfies the multiple inequality

$$\begin{aligned} f(\alpha a+\beta b) &\le \sum_{i=1}^{n}\lambda_i\frac{\int_c^{x_i}f(x)\,dx}{x_i-c} \le \sum_{j=1}^{m}\kappa_j\frac{\int_c^{y_j}f(y)\,dy}{y_j-c} \\ &\le \alpha f(a)+\beta f(b). \end{aligned} \tag{3.13}$$

Given the previous experience, the refinement of the inequality in formula (3.13) stands as

$$\begin{aligned} f(\alpha a+\beta b) &\leq \sum_{i=1}^{n}\lambda_i f\left(\frac{x_i+c}{2}\right) \leq \sum_{i=1}^{n}\lambda_i \frac{\int_c^{x_i} f(x)\,dx}{x_i-c} \\ &\leq \sum_{j=1}^{m}\kappa_j \frac{\int_c^{y_j} f(y)\,dy}{y_j-c} \leq \sum_{j=1}^{m}\kappa_j \frac{f(y_j)+f(c)}{2} \\ &\leq \alpha f(a)+\beta f(b). \end{aligned} \tag{3.14}$$

The Hermite-Hadamard inequality still occupies the attention of mathematicians. Its applications are generally focused on the classes of functions that are more general than convex, see [11], [14] and [16].

4. Generalizations to higher dimensions

If $a_1,\dots,a_{m+1}\in\mathbb{R}^m$ are points such that the differences $a_1-a_{m+1},\dots,a_m-a_{m+1}$ are linearly independent, then the m-simplex with vertices $a_1,\dots,a_{m+1}$ can be introduced as the set

$$\triangle_{a_1\dots a_{m+1}}=\left\{\sum_{j=1}^{m+1}\alpha_j a_j:\ \alpha_j\in[0,1],\ \sum_{j=1}^{m+1}\alpha_j=1\right\}. \tag{4.1}$$

Thus $\triangle_{a_1\dots a_{m+1}}=\mathrm{conv}\{a_1,\dots,a_{m+1}\}$. By applying the above formula to $(k+1)$–membered subsets of $\{a_1\dots a_{m+1}\}$, we get k–subsimplices of $\triangle_{a_1\dots a_{m+1}}$. Usually, 1-subsimplices are called edges, and $(m–1)$–subsimplices are called facets. If we use the above formula with $\alpha_j\in(0,1)$, then we have the open m-simplex $\triangle^o_{a_1\dots a_{m+1}}$.

Each point $x\in\triangle_{a_1\dots a_{m+1}}$ is represented by the unique $(m+1)$-membered convex combination $\sum_{j=1}^{m+1}\alpha_{xj}a_j$ because the points $a_1-a_{m+1},\dots,a_m-a_{m+1}$ are linearly independent. By using the sets $\triangle_{a_j=x}=\mathrm{conv}\{a_1,\dots,a_{j-1},x,a_{j+1},\dots,a_{m+1}\}$ and the denotation vol_m for the volume in the space $\mathbb{R}^m$, the convex combination

$$x=\sum_{j=1}^{m+1}\frac{\mathrm{vol}_m(\triangle_{a_j=x})}{\mathrm{vol}_m(\triangle_{a_1\dots a_{m+1}})}a_j$$

indicates that $\alpha_{xj}=\mathrm{vol}_m(\triangle_{a_j=x})/\mathrm{vol}_m(\triangle_{a_1\dots a_{m+1}})$. Depending on the position of x, the set $\triangle_{a_j=x}$ appears as one of two volumetric shapes, as an m-simplex in the case $\alpha_{xj}>0$, or as the facet $\triangle_{a_1\dots a_{j-1}a_{j+1}\dots a_{m+1}}$ in the case $\alpha_{xj}=0$.

The generalization of Lemma 2.1 to the m-simplex involves a certain convex combination of its vertices.

Lemma 4.1. *Let $\sum_{i=1}^{n}\lambda_i x_i$ be a convex combination of points $x_i\in\triangle_{a_1\dots a_{m+1}}$, and let $\sum_{j=1}^{m+1}\alpha_j a_j$ be the convex combination of the vertices a_j such that*

$$\sum_{i=1}^{n}\lambda_i x_i=\sum_{j=1}^{m+1}\alpha_j a_j.$$

Then each convex function $f : \triangle_{a_1\dots a_{m+1}} \to \mathbb{R}$ satisfies the double inequality

$$f\Big(\sum_{j=1}^{m+1} \alpha_j a_j\Big) \le \sum_{i=1}^{n} \lambda_i f(x_i) \le \sum_{j=1}^{m+1} \alpha_j f(a_j). \tag{4.2}$$

The generalization of Lemma 2.2 to the m-simplex includes the barycenter of its subset with a positive volume.

Lemma 4.2. *Let $X \subseteq \triangle_{a_1\dots a_{m+1}}$ be a set with $\mathrm{vol}_m(X) > 0$, and let $\sum_{j=1}^{m+1} \alpha_j a_j$ be the convex combination of the vertices a_j such that*

$$\left(\frac{\int_X x_1\, dx_1 \dots dx_m}{\mathrm{vol}_m(X)}, \dots, \frac{\int_X x_m\, dx_1 \dots dx_m}{\mathrm{vol}_m(X)}\right) = \sum_{j=1}^{m+1} \alpha_j a_j.$$

Then each convex function $f : \triangle_{a_1\dots a_{m+1}} \to \mathbb{R}$ satisfies the double inequality

$$f\Big(\sum_{j=1}^{m+1} \alpha_j a_j\Big) \le \frac{\int_X f(x_1, \dots, x_m)\, dx_1 \dots dx_m}{\mathrm{vol}_m(X)} \le \sum_{j=1}^{m+1} \alpha_j f(a_j). \tag{4.3}$$

We can easily generalize the statement of Theorem 3.1, but the proof is more demanding.

Theorem 4.1. *Let $f : \triangle_{a_1\dots a_{m+1}} \to \mathbb{R}$ be a convex function, and let $c = \sum_{j=1}^{m+1} \alpha_j a_j$ be a convex combination of the vertices a_j.*

Then the function $f_c : [0, 1] \to \mathbb{R}$ defined by

$$f_c(t) = \sum_{j=1}^{m+1} \alpha_j f\big(ta_j + (1-t)c\big) \tag{4.4}$$

is nondecreasing and convex.

Proof. The representation of f_c: By emphasizing the c-originating convex combinations

$$a_{jc}(t) = ta_j + (1-t)c, \tag{4.5}$$

we appoint the representation

$$f_c(t) = \sum_{j=1}^{m+1} \alpha_j f(a_{jc}(t)). \tag{4.6}$$

The point $a_{jc}(t)$ belongs to the line segment $\triangle_{ca_j}$ with the endpoints c and a_j. The combinations in formula (4.5) are restricted affine mappings $a_{jc} : [0, 1] \to \triangle_{ca_j}$. If $t > 0$, then it can be proved that the points $a_{jc}(t)$ span the m-simplex

$$\triangle_t = \triangle_{a_{1c}(t)\dots a_{m+1\,c}(t)}. \tag{4.7}$$

The limit case is the singleton $\triangle_0 = \{c\}$, and the given simplex is $\triangle_1$.

The monotonicity of f_c: We rely on Lemma 4.1. Let $t_1, t_2 \in [0, 1]$ be a pair of arguments such that $t_1 < t_2$. Then the convex combination

$$a_{jc}(t_1) = \frac{t_2 - t_1}{t_2} c + \frac{t_1}{t_2} a_{jc}(t_2)$$

indicates that the point $a_{jc}(t_1)$ belongs to the line segment $\triangle_{ca_{jc}(t_2)}$. Therefore we have the inclusion $a_{jc}(t_1) \in \triangle_{t_2}$ for $j = 1, \ldots, m+1$ (in fact this proves $\triangle_{t_1} \subset \triangle_{t_2}$). We also have the equality

$$\sum_{j=1}^{m+1} \alpha_j a_{jc}(t_1) = c = \sum_{j=1}^{m+1} \alpha_j a_{jc}(t_2).$$

Then the application of the right-hand side of the double inequality in formula (4.2) to the above inclusion and equality results as the inequality

$$\sum_{j=1}^{m+1} \alpha_j f(a_{jc}(t_1)) \leq \sum_{j=1}^{m+1} \alpha_j f(a_{jc}(t_2))$$

which affirms the relation $f_c(t_1) \leq f_c(t_2)$. Thus f_c is nondecreasing.

The convexity of f_c: Since the compositions $f \circ a_{jc}$ are convex, their convex combination $f_c = \sum_{j=1}^{m+1} \alpha_j (f \circ a_{jc})$ is definitely convex. $\square$

An elegant approach to the simplices $\triangle_t$ can be realized as follows.

Remark 4.1. Let $t_1, t_2 \in (0, 1]$ be numbers satisfying $t_1 < t_2$, and let $H_{c,t_1/t_2} : \mathbb{R}^m \to \mathbb{R}^m$ be the homothety with the center at c and ratio t_1/t_2 standing as

$$H_{c,t_1/t_2}(x) - c = \frac{t_1}{t_2}(x - c).$$

Then the above homothety satisfies

$$H_{c,t_1/t_2}(a_{jc}(t_2)) = c + \frac{t_1}{t_2}(a_{jc}(t_2) - c) = t_1 a_j + (1 - t_1)c = a_{jc}(t_1)$$

for every $j = 1, \ldots, m+1$, so it maps the vertices of $\triangle_{t_2}$ onto the vertices of $\triangle_{t_1}$. Since a homothety preserves affine (including convex) combinations, it follows that the above homothety maps the simplex $\triangle_{t_2}$ onto the simplex $\triangle_{t_1}$.

The relations $f_c(0) \leq f_c(t) \leq f_c(1)$ give the following inequality.

Corollary 4.1. *Let $c = \sum_{j=1}^{m+1} \alpha_j a_j$ be a convex combination of the vertices a_j, and let $t \in [0, 1]$ be a number.*

Then each convex function $f : \triangle_{a_1 \ldots a_{m+1}} \to \mathbb{R}$ satisfies the double inequality

$$f\left(\sum_{j=1}^{m+1} \alpha_j a_j\right) \leq \sum_{j=1}^{m+1} \alpha_j f(t a_j + (1 - t)c) \leq \sum_{j=1}^{m+1} \alpha_j f(a_j). \tag{4.8}$$

The refinement of the inequality in formula (4.8) can be obtained by using any convex combination of points belonging to the simplex $\triangle_{a_1 \ldots a_{m+1}}$.

Theorem 4.2. *Let $c = \sum_{i=1}^{n} \lambda_i x_i$ be a convex combination of points $x_i \in \triangle_{a_1 \ldots a_{m+1}}$, let $c = \sum_{j=1}^{m+1} \alpha_j a_j$ be the convex combination of the vertices a_j, and let $t \in [0, 1]$ be a number.*

Then each convex function $f : \triangle_{a_1 \ldots a_{m+1}} \to \mathbb{R}$ satisfies the multiple inequality

$$\begin{aligned} f\Big(\sum_{j=1}^{m+1} \alpha_j a_j\Big) \ &\leq \ \sum_{i=1}^{n} \lambda_i f(tx_i+(1-t)c) \ \leq \ \sum_{j=1}^{m+1} \alpha_j f(ta_j + (1-t)c) \\ &\leq \ \sum_{j=1}^{m+1} \alpha_j f(a_j). \end{aligned} \tag{4.9}$$

Proof. The inequality of the last two members is contained in formula (4.8).

To ensure the inequality of the first three members, we utilize Lemma 4.1. Given the point x_i, we use the convex combination $x_i = \sum_{j=1}^{m+1} \alpha_{ij} a_j$. Then we have the similar convex combination

$$x_{ic}(t) = \sum_{j=1}^{m+1} \alpha_{ij} a_{jc}(t).$$

It implies the inclusion $x_{ic}(t) \in \triangle_t$ for $i = 1, \ldots, n$. Furthermore, there follows the equality

$$\sum_{i=1}^{n} \lambda_i x_{ic}(t) = c = \sum_{j=1}^{m+1} \alpha_j a_{jc}(t).$$

By applying the double inequality in formula (4.2) to the above inclusion and equality, we get the inequality

$$f\Big(\sum_{j=1}^{m+1} \alpha_j a_{jc}(t)\Big) \leq \sum_{i=1}^{n} \lambda_i f(x_{ic}(t)) \leq \sum_{j=1}^{m+1} \alpha_j f(a_{jc}(t))$$

which ensures the inequality of the first three members. $\square$

The point $\bar{c} = \sum_{j=1}^{m+1} a_j/(m+1)$ is the barycenter of the m-simplex $\triangle_{a_1 \ldots a_{m+1}}$. Within Lemma 4.2, the next corollary exploits the point $\bar{c}$ as the barycenter of m-simplices $\bar{\triangle}_t = \triangle_{a_{1\bar{c}}(t) \ldots a_{m+1\,\bar{c}}(t)}$ with $t > 0$. Obviously, $\triangle_{a_1 \ldots a_{m+1}} = \bar{\triangle}_1$.

Corollary 4.2. *Let $\bar{c} = \sum_{j=1}^{m+1} a_j/(m+1)$ be the barycentric convex combination of the vertices a_j, and let $t \in (0, 1]$ be a number.*

Then each convex function $f : \triangle_{a_1 \ldots a_{m+1}} \to \mathbb{R}$ satisfies the double inequality

$$\begin{aligned} f\Big(\frac{\sum_{j=1}^{m+1} a_j}{m+1}\Big) &\leq \frac{\int_{\bar{\triangle}_t} f(x_1, \ldots, x_m)\, dx_1 \ldots dx_m}{\mathrm{vol}_m(\bar{\triangle}_t)} \leq \frac{\sum_{j=1}^{m+1} f(ta_j+(1-t)\bar{c})}{m+1} \\ &\leq \frac{\sum_{j=1}^{m+1} f(a_j)}{m+1}. \end{aligned} \tag{4.10}$$

Proof. To obtain the inequality of the first three members, we apply formula (4.3) to the m-simplex $X = \bar{\triangle}_t$. The barycentric equality

$$\frac{\sum_{j=1}^{m+1} a_{j\bar{c}}(t)}{m+1} = \frac{\sum_{j=1}^{m+1} a_j}{m+1}$$

plays a role. $\square$

The second member in formula (4.10) approaches $f(\bar{c})$ as t approaches 0. Namely, the first member is the constant $f(\bar{c})$, and the third member approaches $f(\bar{c})$.

By using $t = 1$ in formula (4.10), or using $X = \triangle_{a_1 \ldots a_{m+1}}$ in formula (4.3), we get the Hermite-Hadamard inequality for the m-simplex as

$$f\left(\frac{\sum_{j=1}^{m+1} a_j}{m+1}\right) \leq \frac{\int_{\triangle_{a_1 \ldots a_{m+1}}} f(x_1, \ldots, x_m)\, dx_1 \ldots dx_m}{\operatorname{vol}_m(\triangle_{a_1 \ldots a_{m+1}})} \leq \frac{\sum_{j=1}^{m+1} f(a_j)}{m+1}. \tag{4.11}$$

The Hermite-Hadamard inequality for the simplices was considered in the papers [1, 9, 10, 12, 15] and many others.

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