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SECOND HANKEL DETERMINANT PROBLEM FOR A CERTAIN SUBCLASS OF ANALYTIC BI-UNIVALENT FUNCTIONS

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ABSTRACT. In the present article, we obtain an upper bound for the second Hankel determinant $H_2(2)$ for a certain subclass of analytic bi-univalent functions in the unit disc $\mathbb{U}$. We also give the upper bounds for $H_2(2)$ of some certain sublasses of analytic bi-univalent functions as speial cases of our results.

1. Introduction

Let $\mathbb{U}$ be the open unit disk $\{z : z \in \mathbb{C} \text{ and } |z| < 1\}$ and let $\mathcal{A}$ denote the class of functions analytic in $\mathbb{U}$, satisfying the conditions

$$f(0) = 0 \quad \text{and} \quad f'(0) = 1. \tag{1.1}$$

Then each function $f \in \mathcal{A}$ has the Taylor expansion

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.2}$$

Let $\mathcal{S}$ denote the class of analytic and univalent functions in $\mathbb{U}$ with the normalization conditions (1.1). According to Koebe-One-Quarter Theorem [6, p. 259], every $f \in \mathcal{S}$ has an inverse function f^{-1} satisfying

$$f^{-1}(f(z)) = z \quad (z \in \mathbb{U})$$

and

$$f^{-1}(f(w)) = w \quad \left(|w| < r_0(f);\ r_0(f) \geq \frac{1}{4}\right),$$

where

$$f^{-1}(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \cdots . \tag{1.3}$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{U}$ if both f and f^{-1} are univalent in $\mathbb{U}$. Let Σ denote the class of bi-univalent functions in $\mathbb{U}$ given by (1.2). In the recent articles, various subclasses of bi-univalent functions were investigated (see, for example, [3,4,8,21,25,

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26,29,30]). Generally the upper bounds for the first two coefficient estimates were obtained in these articles. There are only a few works determining the general coefficient bounds $|a_n|$ for the analytic bi-univalent functions (see, for example, [2,10,15]). However, the problem to find the coefficient bounds of $|a_n|$ $(n = 2, 3, 4, \dots)$ for functions $f \in \Sigma$ is still an open problem.

The q^{th} determinant for $q \geq 1$ and $n \geq 0$ is stated by Noonan and Thomas [20] as

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q+1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix} \qquad (a_1 = 1). \tag{1.4}$$

This determinant has also been considered by several authors. For example, Noor [19] determined the rate of growth of $H_q(n)$ as $n \to \infty$ for functions f given by (1.2) with bounded boundary. Ehrenborg [7] studied the Hankel determinant of exponential polynomials. The Hankel transform of an integer sequence and some of its properties were discussed by Layman [16].

Note that

$$H_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_2 a_4 - a_3^2.$$

The determinant $H_2(2)$ is called second Hankel determinant. Many authors have obtained upper bounds for the second Hankel determinant for functions belonging to several subclasses of analytic functions given by (1.2), (see, for example, [11–14, 17, 24, 27, 28]). Recently, the upper bounds of $H_2(2)$ for functions in certain subclasses of Σ have been discussed by many authors (see, for example, [5, 22, 23]).

In the present article, we obtain an upper bound for the second Hankel determinant $H_2(2)$ of a function $f \in \mathcal{A}$, given by (1.2), belongs to a certain subclass of Σ defined by the following:

Definition 1.1. (see [18]) A function $f \in \Sigma$ given by (1.2) is said to be in the class $\mathcal{M}_\Sigma(\beta, \lambda)$ if the following conditions are satisfied:

$$\Re\left(\frac{zf'(z)}{(1-\lambda) f(z) + \lambda z f'(z)}\right) > \beta, \tag{1.5}$$

and

$$\Re\left(\frac{wg'(w)}{(1-\lambda) g(w) + \lambda w g'(w)}\right) > \beta, \tag{1.6}$$

where $0 \leq \beta < 1$; $0 \leq \lambda < 1$; $z, w \in \mathbb{U}$; and $g = f^{-1}$ is given by (1.3).

Remark 1.1. Note that we have the following classes:

(**i**) $\mathcal{M}_\Sigma(\beta, 0) = \mathcal{S}_\Sigma^*(\beta)$ the class of bi-starlike functions of order β $(0 \leq \beta < 1)$, introduced and studied by Brannan and Taha [1],

(**ii**) $\mathcal{M}_\Sigma(0, 0) = \mathcal{S}_\Sigma^*$ the class of bi-starlike functions.

Let $\mathcal{P}$ be the family of all functions p analytic in $\mathbb{U}$ for which $\Re(p(z)) > 0$ and

$$p(z) = 1 + c_1 z + c_2 z + \cdots. \tag{1.7}$$

The following lemmas are required to prove our main results.

Lemma 1.1. [6] *If the function* $p \in \mathcal{P}$ *is given by the series* (1.7)*, then the sharp estimate* $|c_k| \leq 2$ $(k \in \mathbb{N} = \{1, 2, \ldots\})$ *holds.*

Lemma 1.2. [9] *If the function* $p \in \mathcal{P}$ *is given by the series* (1.7)*, then*

$$2c_2 = c_1^2 + x\left(4 - c_1^2\right) \tag{1.8}$$

$$4c_3 = c_1^3 + 2c_1\left(4 - c_1^2\right)x - c_1\left(4 - c_1^2\right)x^2 + 2\left(4 - c_1^2\right)\left(1 - |x|^2\right)z \tag{1.9}$$

for some x, z *with* $|x| \leq 1$, $|z| \leq 1$.

2. Main Results

Theorem 2.1. *Let the function* f *given by* (1.2) *be in the class in* $\mathcal{M}_\Sigma(\beta, \lambda)$*. Then*

$$\left|a_2a_4 - a_3^2\right| \leq \frac{(1-\beta)^2}{(1-\lambda)^2}\begin{cases} \frac{4}{3}\left(4\beta^2 - 8\beta + 5\right) & , \quad \beta \in [0,\, \tau] \\ 1 - \frac{3(1+\lambda)(2-\lambda-\beta)^2}{(1-\lambda)\left[16(1-\lambda)(1-\beta)^2 - 6(1-\beta) - 5(1-\lambda)\right]} & , \quad \beta \in (\tau,\, 1) \end{cases},$$

where

$$\tau = \frac{(29 - 32\lambda) - \sqrt{128\lambda^2 - 256\lambda + 137}}{32(1-\lambda)}.$$

Proof. Since $f \in \mathcal{M}_\Sigma(\beta, \lambda)$, there exists analytic functions $p, q \in \mathcal{P}$ in the unit disk $\mathbb{U}$ with

$$p(0) = 1, \quad \Re(p(z)) > 0$$

and

$$q(0) = 1, \quad \Re(q(z)) > 0$$

such that

$$\frac{zf'(z)}{(1-\lambda)f(z) + \lambda zf'(z)} = \beta + (1-\beta)p(z) \tag{2.1}$$

and

$$\frac{wg'(w)}{(1-\lambda)g(w) + \lambda wg'(w)} = \beta + (1-\beta)q(w) \tag{2.2}$$

for some $z, w \in \mathbb{U}$. Here p and q have the following series expansion

$$p(z) = 1 + c_1z + c_2z^2 + \cdots \tag{2.3}$$

and

$$q(w) = 1 + d_1w + d_2w^2 + \cdots, \tag{2.4}$$

respectively. By using (2.1), (2.2), (2.3) and (2.4), it is obtained that

$$(1-\lambda)a_2 = (1-\beta)c_1, \tag{2.5}$$

$$2(1-\lambda)a_3 - \left(1-\lambda^2\right)a_2^2 = (1-\beta)c_2, \tag{2.6}$$

$$3(1-\lambda)a_4 - (1-\lambda)(3+4\lambda)a_2a_3 + (1-\lambda)(1+\lambda)^2a_2^3 = (1-\beta)c_3, \tag{2.7}$$

and

$$-(1-\lambda)a_2 = (1-\beta)d_1, \tag{2.8}$$

$$-2(1-\lambda)a_3+(1-\lambda)(3-\lambda)a_2^2=(1-\beta)d_2, \tag{2.9}$$

$$-3(1-\lambda)a_4+4(1-\lambda)(3-\lambda)a_2a_3-(1-\lambda)\left(\lambda^2-6\lambda+10\right)a_2^3=(1-\beta)d_3. \tag{2.10}$$

From (2.5) and (2.8), it is obvious that

$$c_1=-d_1 \tag{2.11}$$

and

$$a_2=\frac{(1-\beta)}{(1-\lambda)}c_1. \tag{2.12}$$

By using (2.6), (2.9) and (2.12), we also obtain that

$$a_3=\frac{(1-\beta)^2c_1^2}{(1-\lambda)^2}+\frac{(1-\beta)(c_2-d_2)}{4(1-\lambda)}. \tag{2.13}$$

Finally from (2.7) and (2.10), we get

$$a_4=\frac{2+2\lambda-\lambda^2}{3}\frac{(1-\beta)^3}{(1-\lambda)^3}c_1^3+\frac{5}{8}\frac{(1-\beta)^2}{(1-\lambda)^2}c_1(c_2-d_2)+\frac{1-\beta}{6(1-\lambda)}(c_3-d_3). \tag{2.14}$$

Hence, we can easily write

$$\begin{aligned}\left|a_2a_4-a_3^2\right| &= \left|-\frac{1}{3}\frac{(1-\beta)^4}{(1-\lambda)^2}c_1^4+\frac{(1-\beta)^3}{8(1-\lambda)^3}c_1^2(c_2-d_2)\right.\\ &\quad\left.+\frac{(1-\beta)^2}{6(1-\lambda)^2}c_1(c_3-d_3)-\frac{(1-\beta)^2}{16(1-\lambda)^2}(c_2-d_2)^2\right|.\end{aligned} \tag{2.15}$$

According to Lemma 1.2 and (2.11), we may write

$$c_2-d_2=\frac{4-c_1^2}{2}(x-y) \tag{2.16}$$

and

$$\begin{aligned}&c_3-d_3 \\ &=\frac{c_1^3}{2}+\frac{c_1(4-c_1^2)(x+y)}{2}-\frac{c_1(4-c_1^2)(x^2+y^2)}{4}+\frac{(4-c_1^2)\left[\left(1-|x|^2\right)z-\left(1-|y|^2\right)w\right]}{2}\end{aligned} \tag{2.17}$$

for some x, y, z and w with $|x|\le 1$, $|y|\le 1$, $|z|\le 1$ and $|w|\le 1$. Using (2.16) and (2.17) in (2.15), we have

$$\begin{aligned}\left|a_2a_4-a_3^2\right| &= \left|-\frac{1}{3}\frac{(1-\beta)^4}{(1-\lambda)^2}c_1^4+\frac{(1-\beta)^3}{(1-\lambda)^3}\frac{c_1^2(4-c_1^2)}{16}(x-y)\right.\\ &+\frac{(1-\beta)^2}{6(1-\lambda)^2}c_1\left\{\frac{c_1^3}{2}+\frac{c_1(4-c_1^2)}{2}(x+y)-\frac{c_1(4-c_1^2)}{4}\left(x^2+y^2\right)\right.\\ &\left.\left.+\frac{(4-c_1^2)}{2}\left[\left(1-|x|^2\right)z-\left(1-|y|^2\right)w\right]\right\}-\frac{(1-\beta)^2}{(1-\lambda)^2}\frac{(4-c_1^2)^2}{64}(x-y)^2\right|.\end{aligned}$$

Since the function $p(z)$ and $p(e^{i\theta}z)$, $(\theta \in \mathbb{R})$ are in the class $\mathcal{P}$ simultaneously, we assume without loss of generality that $c_1 > 0$. For convenience of notation, we take $c_1 = c$, $c \in [0,2]$. By using the triangle inequality, it is obtained that

$$\begin{aligned}\left|a_2a_4 - a_3^2\right| \leq & \frac{1}{3}\frac{(1-\beta)^4}{(1-\lambda)^2}c^4 + \frac{(1-\beta)^2}{12(1-\lambda)^2}c^4 + \frac{(1-\beta)^2}{6(1-\lambda)^2}c\left(4-c^2\right) \\ & + \frac{(1-\beta)^2 c^2 (4-c^2)}{4(1-\lambda)^2}\left\{\frac{1}{3} + \frac{1-\beta}{4(1-\lambda)}\right\}(|x|+|y|) \\ & + \frac{(1-\beta)^2 c (4-c^2)(c-2)}{24(1-\lambda)^2}\left(|x|^2+|y|^2\right) + \frac{(1-\beta)^2 (4-c^2)^2}{64(1-\lambda)^2}(|x|+|y|)^2 .\end{aligned}$$

Thus, for $\delta = |x| \leq 1$ and $\mu = |y| \leq 1$ we obtain

$$\left|a_2a_4 - a_3^2\right| \leq T_1 + T_2(\delta+\mu) + T_3\left(\delta^2+\mu^2\right) + T_4(\delta+\mu)^2 = F(\delta,\mu), \tag{2.18}$$

where

$$T_1 = T_1(c) = \frac{1}{3}\frac{(1-\beta)^4}{(1-\lambda)^2}c^4 + \frac{(1-\beta)^2}{12(1-\lambda)^2}c^4 + \frac{(1-\beta)^2}{6(1-\lambda)^2}c\left(4-c^2\right) \geq 0,$$

$$T_2 = T_2(c) = \frac{(1-\beta)^2 c^2 (4-c^2)}{4(1-\lambda)^2}\left\{\frac{1}{3} + \frac{1-\beta}{4(1-\lambda)}\right\} \geq 0,$$

$$T_3 = T_3(c) = -\frac{(1-\beta)^2 c (4-c^2)(2-c)}{24(1-\lambda)^2} \leq 0,$$

$$T_4 = T_4(c) = \frac{(1-\beta)^2 (4-c^2)^2}{64(1-\lambda)^2} \geq 0.$$

Now, we have to determine the maximum of $F(\delta,\mu)$ on the closed square $[0,1]\times[0,1]$. We need to examine the cases $c \in (0,2)$, $c = 2$ and $c = 0$. Let

$$\Pi = \{(\delta,\mu) : 0 \leq \delta \leq 1,\ 0 \leq \mu \leq 1\}.$$

We know that T_3 is negative and

$$T_3 + 2T_4 = \frac{(1-\beta)^2 (4-c^2)(2-c)}{48(1-\lambda)^2}$$

is positive for $c \in (0,2)$. Hence, it is obvious that

$$F_{\delta\delta}F_{\mu\mu} - F_{\delta\mu}^2 = 4T_3(T_3+2T_4) < 0.$$

So, it means that the function $F(\delta,\mu)$ cannot have a local maximum in the interior of the closed square Π. Now, we need to compare the boundary values of $F(\delta,\mu)$.

For $\delta = 0$ and $0 \leq \mu \leq 1$ (similarly for $\mu = 0$ and $0 \leq \delta \leq 1$), we obtain

$$F(0,\mu) = G(\mu) = (T_3+T_4)\mu^2 + T_2\mu + T_1.$$

Here, we have to consider the following two cases:

Case 1: Let $T_3+T_4 \geq 0$. In this case it is clear that $G'(\mu)=2\left(T_3+T_4\right)\mu+T_2>0$ for $0<\mu<1$ and any fixed c with $c\in(0,2)$. Therefore $G(\mu)$ is an increasing function. For fixed $c\in(0,2)$, we have

$$\max_{0<\mu<1} G(\mu)=T_1+T_2+T_3+T_4.$$

Case 2: Let $T_3+T_4<0$. Since $T_2+2\left(T_3+T_4\right)\geq 0$ for $0<\mu<1$ and any fixed $c\in[0,2)$, it is clear that $T_2+2\left(T_3+T_4\right)<T_2+2\left(T_3+T_4\right)\mu<T_2$ and so $G'(\mu)>0$. Hence for fixed $c\in(0,2)$, we have

$$\max_{0<\mu<1} G(\mu)=T_1+T_2+T_3+T_4.$$

For $\delta=1$ and $0\leq\mu\leq 1$ (similarly for $\mu=1$ and $0\leq\delta\leq 1$), we obtain

$$F(1,\mu)=H(\mu)=\left(T_3+T_4\right)\mu^2+\left(T_2+2T_4\right)\mu+T_1+T_2+T_3+T_4.$$

Replaying the above cases, we obtain the following equality

$$\max_{0<\mu<1} H(\mu)=T_1+2T_2+2T_3+4T_4.$$

We see that $G(1)\leq H(1)$ for $c\in(0,2)$. So, we have

$$\max F(\delta,\mu)=F(1,1)$$

on the boundary of the closed square Π.

Let $K:(0,2)\to\mathbb{R}$,

$$K(c)=F(1,1)=T_1+2T_2+2T_3+4T_4. \tag{2.19}$$

Substituting the values of T_1, T_2, T_3 and T_4 in the function $K(c)$ yields

$$K(c)=\frac{(1-\beta)^2}{12\left(1-\lambda\right)^2}\left\{\left[4\left(1-\beta\right)^2-\frac{3\left(1-\beta\right)}{2\left(1-\lambda\right)}-\frac{5}{4}\right]c^4+6\left[1+\frac{1-\beta}{1-\lambda}\right]c^2+12\right\}.$$

We need to determine the maximum of $K(c)$. After some elementary calculations, we obtain

$$K'(c)=\frac{(1-\beta)^2 c}{3\left(1-\lambda\right)^2}\left\{\left[4\left(1-\beta\right)^2-\frac{3\left(1-\beta\right)}{2\left(1-\lambda\right)}-\frac{5}{4}\right]c^2+3\left[1+\frac{1-\beta}{1-\lambda}\right]\right\}. \tag{2.20}$$

Now, we have to do following examine:

Case 1: Let

$$4\left(1-\beta\right)^2-\frac{3\left(1-\beta\right)}{2\left(1-\lambda\right)}-\frac{5}{4}\geq 0.$$

It means that $\beta\in\left[0,\frac{(13-16\lambda)-\sqrt{80\lambda^2-160\lambda+89}}{16(1-\lambda)}\right]$. Hence $K'(c)>0$ for $c\in(0,2)$. It means that it has no maximum value in this interval since $K(c)$ is an increasing function in the interval $(0,2)$.

Case 2: Let

$$4\left(1-\beta\right)^2-\frac{3\left(1-\beta\right)}{2\left(1-\lambda\right)}-\frac{5}{4}<0.$$

It is possible for $\beta \in \left(\frac{(13-16\lambda)-\sqrt{80\lambda^2-160\lambda+89}}{16(1-\lambda)}, 1\right)$. Then we see that the function $K'(c)$ has the critical points

$$c_{01} = 0 \text{ and } c_{02} = \sqrt{\frac{-12(2-\lambda-\beta)}{16(1-\lambda)(1-\beta)^2 - 6(1-\beta) - 5(1-\lambda)}}.$$

If $\beta \in \left(\frac{(13-16\lambda)-\sqrt{80\lambda^2-160\lambda+89}}{16(1-\lambda)}, \frac{(29-32\lambda)-\sqrt{128\lambda^2-256\lambda+137}}{32(1-\lambda)}\right]$, then we observe that $c_{02} \geq 2$. It means that c_{02} is out of the interval $(0,2)$. If $\beta \in \left(\frac{(29-32\lambda)-\sqrt{128\lambda^2-256\lambda+137}}{32(1-\lambda)}, 1\right)$, we see that $c_{02} < 2$. Since $K''(c) < 0$, the function $K(c)$ has a maximum at $c = c_{02}$ which is in the interval $(0,2)$. Hence we have

$$\begin{aligned} \max_{0<c<2} K(c) &= K(c_{02}) \\ &= \frac{(1-\beta)^2}{(1-\lambda)^2}\left\{-\frac{3(1+\lambda)(2-\lambda-\beta)^2}{(1-\lambda)\left[16(1-\lambda)(1-\beta)^2 - 6(1-\beta) - 5(1-\lambda)\right]} + 1\right\}. \end{aligned} \tag{2.21}$$

On the other hand, in the second case for $c = 2$ and $(\delta, \mu) \in \Pi$, we obtain

$$F(\delta,\mu) = \frac{4(1-\beta)^2}{3(1-\lambda)^2}\left(4\beta^2 - 8\beta + 5\right) \tag{2.22}$$

for $\beta \in [0,1)$ and $\lambda \in [0,1)$.

Finally, for $c = 0$ and $(\delta, \mu) \in \Pi$, we have

$$F(\delta,\mu) = \frac{(1-\beta)^2}{4(1-\lambda)^2}(\delta+\mu)^2, \tag{2.23}$$

for $\beta \in [0,1)$ and $\lambda \in [0,1)$. From (2.21), (2.22) and (2.23), it is obvious that

$$\begin{aligned} \frac{(1-\beta)^2}{(1-\lambda)^2} &< \frac{4(1-\beta)^2}{3(1-\lambda)^2}\left(4\beta^2 - 8\beta + 5\right) \\ &< \frac{(1-\beta)^2}{(1-\lambda)^2}\left\{-\frac{3(1+\lambda)(2-\lambda-\beta)^2}{(1-\lambda)\left[16(1-\lambda)(1-\beta)^2 - 6(1-\beta) - 5(1-\lambda)\right]} + 1\right\} \end{aligned}$$

for $\beta \in \left(\frac{(29-32\lambda)-\sqrt{128\lambda^2-256\lambda+137}}{32(1-\lambda)}, 1\right)$. We see that our second inequality holds. On the other hand, we obtain

$$\frac{(1-\beta)^2}{(1-\lambda)^2} < \frac{4(1-\beta)^2}{3(1-\lambda)^2}\left(4\beta^2 - 8\beta + 5\right)$$

for every $\beta \in [0,1)$. Thus we have our first inequality holds for
$\beta \in \left[0, \frac{(29-32\lambda)-\sqrt{128\lambda^2-256\lambda+137}}{32(1-\lambda)}\right]$. The proof is completed. □

We obtain the following corollaries as a special cases of our parameters.

Taking $\lambda = 0$ in Theorem 2.1, the following result is obtained for bi-starlike functions of order β $(0 \leq \beta < 1)$.

Corollary 2.1. *(see* [5, Theorem 2.1]*) Let $f(z)$ given by* (1.2) *be in the class $\mathcal{S}^*_\Sigma(\beta)$. Then*

$$\left|a_2a_4 - a_3^2\right| \leq \begin{cases} \frac{4}{3}(1-\beta)^2\left(4\beta^2-8\beta+5\right) & , \quad \beta \in \left[0, \frac{29-\sqrt{137}}{32}\right] \\ (1-\beta)^2\left(\frac{13\beta^2-14\beta-7}{16\beta^2-26\beta+5}\right) & , \quad \beta \in \left(\frac{29-\sqrt{137}}{32}, 1\right) \end{cases}.$$

Taking $\beta = 0$ and $\lambda = 0$ in Theorem 2.1 yields the following coefficient estimates for bi-starlike functions.

Corollary 2.2. *Let $f(z)$ given by* (1.2) *be in the class S^*_Σ. Then*

$$\left|a_2a_4 - a_3^2\right| \leq \frac{20}{3}.$$

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