Turkish J. Ineq., 3 (1) (2019), Pages 35 –53 .

Turkish Journal of
INEQUALITIES

Available online at www.tjinequality.com

A TWO-PARAMETER SINGULAR FRACTIONAL DIFFERENTIAL EQUATION OF LANE EMDEN TYPE

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Abstract. This paper deals with a nonlinear singular differential equation of Lane Emden type involving Caputo fractional derivatives with two different parameters. We study the existence and uniqueness of solutions for the considered problem. Then, we generate some results for the stability in the sense of Ulam-Hyers.

1. Introduction

The fractional differential equations theory is an important tool in modeling many real world phenomena since it can describe better many processes in biology, signal and image processing, cosmology, physics, chemistry, etc... For more details, we refer the reader to [5–8, 13, 14, 16, 18, 20, 24, 26–28]. In particular, two important and interesting areas of research of this fractional theory are devoted, in one hand, to the study of singular fractional differential equations [1, 2, 21, 22, 25] and, in the other hand, to investigate the Ulam-Hyers stability phenomena in the singular theory (UH stability for short), for more details, see the papers [4, 9, 21–23].
In the present work, we are motivated by the papers in [3, 10–12, 15, 17, 19] that are articulated around the singular Lane Emden problem (L-E problem).
We note that the classical L-E model is given by equation:

$$x^{''}(t)+\frac{a}{t}x^{'}(t)+f(t,x(t))=g(t),\ t\in(0,1],$$

with the conditions

$$x(0)=A\in\mathbb{R},\ x^{'}(0)=B\in\mathbb{R},$$

and f, g are given continuous functions.
With regard to the above equation, the interested reader can find many research papers

Key words and phrases. Caputo derivative, singular differential equation existence and uniqueness, Ulam-Hyers stability.
2010 *Mathematics Subject Classification.* Primary: 26D15. Secondary: 26A33, 60E15.
Received: 02/02/2019
Accepted: 21/06/2019.

that have studied the L-E model. To cite a few, we begin by the above cited reference [15] where the authors proposed a numerical approach to study the problem:

$$\begin{cases} D^{\alpha}y(t)+\dfrac{k}{t^{\alpha-\beta}}D^{\beta}y(t)+f(t,y(t))=g(t),\ t\in(0,1], \\ \\ k\geq 0,\ 1<\alpha\leq 2,\ 0<\beta\leq 1, \end{cases}$$

with

$$y(0)=A,\ y^{'}(0)=B,$$

and $A,B\in\mathbb{R}$.
Also, we recall that in [11], R.W. Ibrahim has been concerned with the following equation with its UH stability:

$$\begin{cases} D^{\beta}\left(D^{\alpha}+\frac{a}{t}\right)u(t)+f(t,u(t))=g(t), \\ u(0)=\mu,\ u(1)=\nu, \\ 0<\alpha,\beta\leq 1,\ 0<t\leq 1,\ a\geq 0, \end{cases}$$

such that D^{γ} is the Caputo derivative; $\gamma>0$, the function f is continuous and $g\in C([0,1])$.
In [23], the authors investigated the following problem:

$$\begin{cases} D^{\beta_1}\left(D^{\alpha_1}+\frac{a_1}{t}\right)x_1(t)+f_1(t,x_1(t),x_2(t),...,x_n(t))=g_1(t),\ t\in(0,1], \\ D^{\beta_2}\left(D^{\alpha_2}+\frac{a_2}{t}\right)x_2(t)+f_2(t,x_1(t),x_2(t),...,x_n(t))=g_2(t),\ t\in(0,1], \\ \vdots \\ D^{\beta_n}\left(D^{\alpha_n}+\frac{a_n}{t}\right)x_n(t)+f_n(t,x_1(t),x_2(t),...,x_n(t))=g_n(t),\ t\in(0,1], \\ \sum\limits_{k=1}^{n}|x_k(0)|=\sum\limits_{k=1}^{n}\left|x_k^{'}(0)\right|=...=\sum\limits_{k=1}^{n}\left|x_k^{(l-1)}(0)\right|=0, \\ \sum\limits_{k=1}^{n}|D^{\alpha_k}x_k(0)|=\sum\limits_{k=1}^{n}|D^{\alpha_{k+1}}x_k(0)|=...=\sum\limits_{k=1}^{n}|D^{\alpha_{k+l-2}}x_k(0)|=0, \\ D^{\alpha_{k+l-1}}x_k(1)=0,\ k=1,2,...,n, \end{cases}$$

such that $l-1<\alpha_k,\beta_k<l,\ a_k\geq 0,\ l\in\mathbb{N}-\{0,1\},\ k=1,2,...,n,\ n\in\mathbb{N}-\{0\}$.

In the recent paper [4], Z. Dahmani and M.Z. Sarikaya introduced the $\Delta-$Ulam stability for the system:

$$\begin{cases} D^{\beta_1}(D^{\alpha_1}+b_1g_1(t))x_1(t)+f_1(t,x_1(t),x_2(t))=h_1(t),\ 0<t<1, \\ \\ D^{\beta_2}(D^{\alpha_2}+b_2g_2(t))x_2(t)+f_2(t,x_1(t),x_2(t))=h_2(t),\ 0<t<1, \\ \\ x_k(0)=0,\ D^{\alpha}x_k(1)+b_kg_k(1)x_k(1)=0,\ k=1,2, \end{cases}$$

with the conditions $0<\alpha_k,\beta_k<1$, $b_k\geq 0$, $k=1,2$. The derivatives D^{β_k} and D^{α_k}, $k=1,2$, are in the sense of Caputo.
In the very recent work of Z. Bekkouche et al. [3], it has been studied the following $2D-$system:

$$\begin{cases} D^{\beta_1}(D^{\alpha_1}+b_1g_1(t))x_1(t)+f_1(t,x_1(t),x_2(t))=\omega_1S_1(t,x_1(t),x_2(t)),\ 0<t<1, \\ D^{\beta_2}(D^{\alpha_2}+b_2g_2(t))x_2(t)+f_2(t,x_1(t),x_2(t))=\omega_2S_2(t,x_1(t),x_2(t)),\ 0<t<1, \\ x_k(0)=0, D^{\alpha}x_k(1)+b_kg_k(1)x_k(1)=0, \end{cases}$$

under the conditions $0 < \beta_k < 1, 0 < \alpha_k < 1, b_k \geq 0, 0 < \omega_k < \infty, k = 1,2$ and the derivatives D^{β_k} and D^{α_k} are of Caputo. The given functions $f_k : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ and $S_k : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ are continuous, $g_k :]0,1] \to [0,+\infty)$ is continuous and singular at $t = 0$.

The present work deals with a more complicated singular fractional equation of Lane Emden type. By introducing the derivative of Caputo in both sides of the equation and by considering α and β, two parameters of derivation without commutativity properties, we study the following problem:

$$\begin{cases} D^{\beta}(D^{\alpha} + \dfrac{k}{t^{\alpha-\beta}})y(t) + \lambda f(t, D^{\delta}y(t)) + g(t, y(t)) = h\,(t)\,, \\ y\,(0) = m,\ y\,(1) = m^*, \\ 0 < \beta < \alpha < 1, t \in]0,1[,\ k \geq 0,\ \lambda \in \mathbb{R} \end{cases} \tag{1.1}$$

For (1), we take $J := [0,1]$, the derivative D^{α} is taken in the sense of Caputo, the functions $f, g : J \times \mathbb{R} \to \mathbb{R}$ are continuous and h is continuous over J.
The main advantage of using Caputo derivative and not the Riemann-Liouville one is that when using Caputo approach in differential equations, standard initial conditions of derivative in term of integer order are involved, but with Riemann-Liouville approach, such conditions are not allowed. These initial conditions have a clear physical interpretation. Another advantage of Caputo approach is that it requires the existence of the $n-$th derivative of the unknown functions, and we know that most functions that appear in applications fulfill this requirement.

2. Preliminaries on Fractional Calculus

We recall some definitions and lemmas that will be used later. For more details, we refer to [16].

Definition 2.1. Let $\alpha > 0$, and $f : [0,1] \longmapsto \mathbb{R}$ be a continuous function.The Riemann-Liouville integral of order α is defined by:

$$I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau,\ \alpha > 0\ ,\ 0 < t < 1.$$

where $\Gamma(\alpha) := \int_0^{\infty} e^{-u} u^{\alpha-1} du$

Definition 2.2. For a function $h \in C^n([0,1], \mathbb{R})$ and $n-1 < \alpha \leq n$, the Caputo fractional derivative is defined by:

$$\begin{aligned} D^{\alpha} h(t) &= I^{n-\alpha} \frac{d^n}{dt^n}(h(t)) \\ &= \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} h^{(n)}(s) ds. \end{aligned}$$

In order to study the problem (1), we need the following two lemmas:

Lemma 2.1. *Let $n \in \mathbb{N}^*$, and $n-1 < \alpha < n$. The general solution of $D^\alpha y(t) = 0$ is given by*

$$y(t) = c_0 + c_1 t + c_2 t^2 + ... + c_{n-1} t^{n-1} \tag{2.1}$$

where $c_i \in \mathbb{R}, i = 0, 1, 2, .., n-1$.

Lemma 2.2. *Given $n \in \mathbb{N}^*$, and $n-1 < \alpha < n$. Then*

$$I^\alpha D^\alpha y(t) = y(t) + c_0 + c_1 t + c_2 t^2 + ... + c_{n-1} t^{n-1} \tag{2.2}$$

for some $c_i \in \mathbb{R}, i = 0, 1, 2, .., n-1$.

We need also to prove the following integral representation:

Lemma 2.3. *Let $G \in C([0,1])$. Then, the problem*

$$\begin{cases} D^\beta (D^\alpha + \frac{k}{t^{\alpha-\beta}}) y(t) = G(t), t \in]0,1[\\ y(0) = m, \ y(1) = m^*, \\ 0 < \beta < \alpha < 1, \ k \geq 0, \ \lambda \in \mathbb{R} \end{cases} \tag{2.3}$$

admits as solution the function:

$$\begin{aligned} y(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \Big(\int_0^s \frac{(s-u)^{\beta-1}}{\Gamma(\beta)} G(u) du - \frac{k}{s^{\alpha-\beta}} y(s) \Big) ds \\ &- \frac{t^\alpha}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} \Big(\int_0^s \frac{(s-u)^{\beta-1}}{\Gamma(\beta)} G(u) du - \frac{k}{s^{\alpha-\beta}} y(s) \Big) ds \\ &- t^\alpha (m - m^*) + m. \end{aligned} \tag{2.4}$$

Proof. By using lemmas 3 and 4, we can written as:

$$\begin{aligned} y(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \Big(\int_0^s \frac{(s-u)^{\beta-1}}{\Gamma(\beta)} G(u) du - \frac{k}{s^{\alpha-\beta}} y(s) \Big) ds \\ &- c_0 I^\alpha(1) - c_1 \end{aligned} \tag{2.5}$$

Then, it yields that

$$\begin{aligned} c_1 &= -m \\ c_0 &= \frac{\Gamma(\alpha+1)}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} \Big(\int_0^s \frac{(s-u)^{\beta-1}}{\Gamma(\beta)} G(u) du - \frac{k}{s^{\alpha-\beta}} y(s) \Big) ds \\ &+ \Gamma(\alpha+1)(m - m^*). \end{aligned} \tag{2.6}$$

Replacing c_0, c_1 in (2.5), we get the desired formule (2.4).

□

Now, let us introduce the Banach space:

$$X := \{y \in C(J,\mathbb{R}), D^{\delta} y \in C(J,\mathbb{R})\}$$

and the norm:

$$\|x\|_X = Max\{\|x\|_\infty \ , \|D^{\delta}x\|_\infty\},$$

where,

$$\|x\|_\infty = \sup_{t\in J} |x(t)| \ , \|D^{\delta}x\|_\infty = \sup_{t\in J} |D^{\delta}x(t)|.$$

Then, we define the nonlinear operator $\mathcal{H}$: $X \to X$ as follows:

$$\begin{aligned}
\mathcal{H}y(t) \quad = \quad & \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\Big(\int_0^s \frac{(s-u)^{\beta-1}}{\Gamma(\beta)}(h(u) \\
& -\lambda f(u, D^{\delta}y(u)) - g(u,y(u)))du - \frac{k}{s^{\alpha-\beta}}y(s)\Big)ds \\
& -\frac{t^{\alpha}}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\Big(\int_0^s \frac{(s-u)^{\beta-1}}{\Gamma(\beta)} \\
& \times(h(u) - \lambda f(u, D^{\delta}y(u)) - g(u,y(u)))du - \frac{k}{s^{\alpha-\beta}}y(s)\Big)ds \\
& -t^{\alpha}(m-m^*) + m.
\end{aligned} \tag{2.7}$$

Now, we are ready to study the above problem by means of the fixed point theory.

3. Main Results

3.1. **A Unique Solution.** We introduce the following quantities:

$$\begin{aligned}
\Omega_{\alpha,\beta,\delta} \quad &:= \quad \frac{1}{\Gamma(\beta+\alpha-\delta+1)} + \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+\beta+1)\Gamma(\alpha-\delta+1)} \\
\Omega'_{\alpha,\beta,\delta} \quad &:= \quad \frac{\Gamma(\alpha+1)\Gamma(\beta-\alpha+1)}{\Gamma(\alpha-\delta+1)\Gamma(\beta+1)} + \frac{\Gamma(\beta-\alpha+1)}{\Gamma(\beta-\delta+1)},
\end{aligned}$$

and

$$d =: Max \begin{cases} 2\frac{(\beta+1,\alpha)}{\Gamma(\alpha)\Gamma(\beta+1)}\Big(|\lambda|L_f + L_g + k\Gamma(\alpha)\Gamma(\beta-\alpha+1)\Big), \\ \Big(\frac{|\lambda|L_f}{\Gamma(\alpha+\beta-\delta+1)} + \frac{\beta\ (\beta+1,\alpha)|\lambda|L_f}{\Gamma(\alpha-\delta+1)} + \frac{L_g}{\Gamma(\alpha+\beta-\delta+1)} \\ +\frac{\beta\ (\beta+1,\alpha)L_g}{\Gamma(\alpha-\delta+1)} + K\ \frac{\Gamma(\beta-\alpha+1)}{\Gamma(\beta-\delta+1)} + k\frac{\alpha\beta\ (\beta-\alpha+1,\alpha)}{\Gamma(\alpha-\delta+1)}\Big) \end{cases}$$

Also, we consider the following hypotheses:

$(H1)$: The functions f, g : $J \times \mathbb{R} \to \mathbb{R}$ are continuous.

$(H2)$: There exist nonnegative constants L_f and L_g such that for all

t $\in J$, $(x, y) \in \mathbb{R}^2$,

$$\begin{array}{rcl} |f(t,x) - f(t,y)| & \leq & L_f|x-y| \\ |g(t,x) - g(t,y)| & \leq & L_g|x-y|. \end{array} \tag{3.1}$$

$(H3)$: There exists non negative constants M_f, M_g such that for all t $\in J$, $x \in \mathbb{R}$, we have

$$|f(t,x)| \quad \leq M_f, |g(t,x)| \quad \leq M_g.$$

$(H4)$: There exists non negative constant M_h that satisfies: $|h(t)| \leq M_h$, for all t $\in J$.

Now we are ready to prove the following result:

Theorem 3.1. *Assume that* $(H2)$,$(H3)$*and* $(H4)$ *hold. Then, the problem (1.1) has a unique solution on* J *provided that* $0 < d < 1$.

Proof. We need to proceed on two steps:

Step1: We consider the set $B_r := \{z \in X; \|z\|_X \leq r,\ r > 0\}$, where:

$$r \geq Max \begin{cases} 2\frac{M_{h,f,g} + (|m| + |m^*|)\Gamma(\beta+\alpha+1)\Gamma(\beta+1)}{\Gamma(\beta+1)(\Gamma(\alpha+\beta+1) - (|\lambda|L_f + L_g) + k\Gamma(\beta-\alpha+1)\Gamma(\alpha+\beta+1)}; \\ \frac{M_{h,f,g}\ \Omega_{\alpha,\beta,\delta}}{1 - (|\lambda|L_f + L_g)\ \Omega_{\alpha,\beta,\delta} - k\Omega'_{\alpha,\beta,\delta}} \\ +\frac{\Gamma(\alpha+1)(m-m^*)}{\Gamma(\alpha-\delta+1)(1 - (|\lambda|L_f + L_g)\ \Omega_{\alpha,\beta,\delta} + k\Omega'_{\alpha,\beta,\delta})}. \end{cases}$$

For $y \in B_r$ and $t \in J$, we have

$$\begin{aligned}
|\mathcal{H}y(t)| &= |\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\Big(\int_0^s \frac{(s-u)^{\beta-1}}{\Gamma(\beta)}(h(u)-\lambda f(u,D^\delta y(u)) \\
&\quad -g(u,y(u)))du - \frac{k}{s^{\alpha-\beta}}y(s)\Big)ds - \frac{t^\alpha}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1} \\
&\quad -\frac{t^\alpha}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\Big(\int_0^s \frac{(s-u)^{\beta-1}}{\Gamma(\beta)}(h(u)-\lambda f(u,D^\delta y(u))) \\
&\quad -g(u,y(u)))du - \frac{k}{s^{\alpha-\beta}}y(s)\Big)ds - t^\alpha(m-m^*) + m|.
\end{aligned} \tag{3.2}$$

Then, we get

$$\begin{aligned}
|\mathcal{H}y(t)| &\le \sup_{t\in J} \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\Big(\int_0^s \frac{(s-u)^{\beta-1}}{\Gamma(\beta)}|(h(u)-\lambda f(u,D^\delta y(u)) \\
&\quad -f(u,0)+f(u,0)) - g(u,y(u)) - g(u,0) + g(u,0)|)du \\
&\quad +|\frac{k}{s^{\alpha-\beta}}y(s)|\Big)ds + \sup_{t\in J}\frac{t^\alpha}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\Big(\int_0^s \frac{(s-u)^{\beta-1}}{\Gamma(\beta)}|(h(u) \\
&\quad -\lambda f(u,D^\delta y(u)) - f(u,0) + f(u,0) - g(u,y(u)) - g(u,0) \\
&\quad +g(u,0)|)du + |\frac{k}{s^{\alpha-\beta}}y(s)|\Big)ds + \sup_{t\in J} t^\alpha|m-m^*| + |m|.
\end{aligned} \tag{3.3}$$

By $(H2)$, $(H3)$and $(H4)$, we have

$$\begin{aligned}
\|\mathcal{H}y\|_\infty &\le 2\Big(M_{h,f,g} + |\lambda|L_f\|D^\delta y\| + L_g\|y\|\Big) \\
&\quad \times\frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\int_0^s \frac{(s-u)^{\beta-1}}{\Gamma(\beta)}duds \\
&\quad +k\|y\|\frac{\Gamma(\beta-\alpha+1)}{\Gamma(\beta+1)} + 2|m| + |m^*|.
\end{aligned}$$

Therefore, we can write

$$\begin{aligned}\|\mathcal{H}y\|_\infty &\leq 2\frac{M_{h,f,g}+(|\lambda|L_f+L_g)r}{\Gamma(\alpha+\beta+1)}\\&\quad+r\frac{k\Gamma(\beta-\alpha+1)}{\Gamma(\beta+1)}+2|m|+|m^*|\end{aligned}\tag{3.4}$$

Moreover, we have:

$$\begin{aligned}D^\delta\mathcal{H}y(t) &= \frac{1}{\Gamma(\alpha-\delta)}\int_0^t(t-s)^{\alpha-\delta-1}\Big(\int_0^s\frac{(s-u)^{\beta-1}}{\Gamma(\beta)}(h(u)\\&\quad-\lambda f(u,D^\delta y(u))-g(u,y(u)))du-\frac{k}{s^{\alpha-\beta}}y(s)\Big)ds\\&\quad-\frac{\alpha\, t^{\alpha-\delta}}{\Gamma(\alpha-\delta+1)}\int_0^1(1-s)^{\alpha-1}\Big(\int_0^s\frac{(s-u)^{\beta-1}}{\Gamma(\beta)}(h(u)\\&\quad-\lambda f(u,D^\delta y(u))-g(u,y(u)))du-\frac{k}{s^{\alpha-\beta}}y(s)\Big)ds\\&\quad-\frac{\Gamma(\alpha+1)t^{\alpha-\delta}}{\Gamma(\alpha-\delta+1)}(m-m^*)\end{aligned}\tag{3.5}$$

Thus,

$$\begin{aligned}\|D^\delta\mathcal{H}y\|_\infty &\leq \Big(M_{h,f,g}+(|\lambda|L_f+L_g)r\Big)\\&\quad\times\Big(\frac{1}{\Gamma(\beta+\alpha-\delta+1)}+\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+\beta+1)\Gamma(\alpha-\delta+1)}\Big)\\&\quad+kr\Big(\frac{\Gamma(\alpha+1)\Gamma(\beta-\alpha+1)}{\Gamma(\alpha-\delta+1)\Gamma(\beta+1)}\\&\quad+\frac{\Gamma(\beta-\alpha+1)}{\Gamma(\beta-\delta+1)}\Big)+\frac{\Gamma(\alpha+1)}{\Gamma(\alpha-\delta+1)}|m-m^*|.\end{aligned}\tag{3.6}$$

Hence, $\|\mathcal{H}y\|_X\leq r$. □

Step 2: We proceed to prove that $\mathcal{H}$ is a contraction mapping.
For $(x,y)\in X^2$ and for each $t\in J$, we can write

$$\begin{aligned}
|\mathcal{H}x(t) - \mathcal{H}y(t)| &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \Big(\int_0^s \frac{(s-u)^{\beta-1}}{\Gamma(\beta)} (h(u) - \lambda f(u, D^\delta x(u)) \\
&- g(u, x(u)) du - \frac{k}{s^{\alpha-\beta}} x(s) \Big) ds - \frac{t^\alpha}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} \\
&\times \Big(\int_0^s \frac{(s-u)^{\beta-1}}{\Gamma(\beta)} (h(u) - \lambda f(u, D^\delta x(u)) - g(u, x(u))) du \\
&- \frac{k}{s^{\alpha-\beta}} x(s) \Big) ds - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \Big(\int_0^s \frac{(s-u)^{\beta-1}}{\Gamma(\beta)} (h(u) \\
&- \lambda f(u, D^\delta y(u)) - g(u, y(u))) du - \frac{k}{s^{\alpha-\beta}} y(s) \Big) ds + \frac{t^\alpha}{\Gamma(\alpha)} \\
&\times \int_0^1 (1-s)^{\alpha-1} \Big(\int_0^s \frac{(s-u)^{\beta-1}}{\Gamma(\beta)} (h(u) - \lambda f(u, D^\delta y(u)) \\
&- g(u, y(u)) du - \frac{k}{s^{\alpha-\beta}} y(s) \Big) ds|. \qquad (3.7)
\end{aligned}$$

We get

$$\begin{aligned}
&|\mathcal{H}x(t) - \mathcal{H}y(t)| \\
&\leq \sup_{t\in J} \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \Big(\int_0^s \frac{(s-u)^{\beta-1}}{\Gamma(\beta)} |\lambda| |f(u, D^\delta x(u)) - f(u, D^\delta y(u))| du \Big) ds \\
&+ \sup_{t\in J} \frac{t^\alpha}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} \Big(\int_0^s \frac{(s-u)^{\beta-1}}{\Gamma(\beta)} |\lambda| |f(u, D^\delta x(u)) - f(u, D^\delta y(u))| du \Big) ds \\
&+ \sup_{t\in J} \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \Big(\int_0^s \frac{(s-u)^{\beta-1}}{\Gamma(\beta)} |g(u, x(u)) - g(u, y(u))| du \Big) ds \qquad (3.8) \\
&+ \sup_{t\in J} \frac{t^\alpha}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} \Big(\int_0^s \frac{(s-u)^{\beta-1}}{\Gamma(\beta)} |g(u, x(u)) - g(u, y(u))| du \Big) ds \\
&+ \sup_{t\in J} \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \frac{k}{s^{\alpha-\beta}} (x(s) - y(s)) ds \\
&+ \sup_{t\in J} \frac{t^\alpha}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} \frac{k}{s^{\alpha-\beta}} (x(s) - y(s)) ds.
\end{aligned}$$

Thanks to $(H2)$, we can write
$\|\mathcal{H}x - \mathcal{H}y\|_\infty$

$$
\begin{aligned}
&\leq \frac{|\lambda|L_f}{\Gamma(\alpha)} \sup_{t\in J} \int_0^t (t-s)^{\alpha-1}\Big(\int_0^s \frac{(s-u)^{\beta-1}}{\Gamma(\beta)} du \|D^\delta x - D^\delta y\| du\Big) ds \\
&+ \sup_{t\in J} \frac{|\lambda|L_f t^\alpha}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1}\Big(\int_0^s \frac{(s-u)^{\beta-1}}{\Gamma(\beta)} \|D^\delta x - D^\delta y\| du\Big) ds \\
&+ \frac{L_g}{\Gamma(\alpha)} \sup_{t\in J} \int_0^t (t-s)^{\alpha-1}\Big(\int_0^s \frac{(s-u)^{\beta-1}}{\Gamma(\beta)} ||x-y|| du\Big) ds \\
&+ \sup_{t\in J} \frac{L_g t^\alpha}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1}\Big(\int_0^s \frac{(s-u)^{\beta-1}}{\Gamma(\beta)} \|x-y\| du\Big) ds \\
&+ \sup_{t\in J} k\|x-y\| I^\alpha t^{\beta-\alpha}(1+t^\alpha).
\end{aligned}
\tag{3.9}
$$

Then,

$$
\begin{aligned}
\|\mathcal{H}x - \mathcal{H}y\|_\infty \;\leq\; & 2|\lambda|L_f \frac{\beta(\beta+1,\alpha)}{\Gamma(\alpha)\Gamma(\beta+1)} \|D^\delta(x-y)\| \\
& + 2L_g \frac{\beta(\beta+1,\alpha)}{\Gamma(\alpha)\Gamma(\beta+1)} ||x-y|| + 2K \frac{\Gamma(\beta-\alpha+1)}{\Gamma(\beta+1)} ||x-y||.
\end{aligned}
\tag{3.10}
$$

Consequently,

$$
\|\mathcal{H}x - \mathcal{H}y\|_\infty \;\leq\; 2\frac{(\beta+1,\alpha)}{\Gamma(\alpha)\Gamma(\beta+1)}\Big(|\lambda|L_f + L_g + k\Gamma(\alpha)\Gamma(\beta-\alpha+1)\Big)\|x-y\|_X. \tag{3.11}
$$

Similarly, we have

$$
\begin{aligned}
\|D^\delta\mathcal{H}x - D^\delta\mathcal{H}y\|_\infty \leq \Big(& \frac{|\lambda|L_f}{\Gamma(\alpha+\beta-\delta+1)} + \frac{\beta\;(\beta+1,\alpha)|\lambda|L_f}{\Gamma(\alpha-\delta+1)} + \frac{L_g}{\Gamma(\alpha+\beta-\delta+1)} \\
& + \frac{\beta\;(\beta+1,\alpha)L_g}{\Gamma(\alpha-\delta+1)} + K\frac{\Gamma(\beta-\alpha+1)}{\Gamma(\beta-\delta+1)} + k\frac{\alpha\beta\;(\beta-\alpha+1,\alpha)}{\Gamma(\alpha-\delta+1)}\Big)\|x-y\|_X.
\end{aligned}
\tag{3.12}
$$

Combining (3.11) and (3.12), it yields that

$$
\|\mathcal{H}x - \mathcal{H}y\|_X \;\leq\; d\,\|x-y\|_X.
$$

3.2. At Least One Solution.

Theorem 3.2. *Assume that hypotheses* $(H1)$*and* $(H3)$ *are fulfilled. Then, the problem* (1.1) *has at least one solution over J..*

Proof. Let us subdivide the proof into the following steps:
Step 1: First of all, we show that the operator $\mathcal{H}$ is completely continuous on X.
We have

$$\begin{aligned}|\mathcal{H}(x_n)(t)-\mathcal{H}(x)(t)| = \Big|&\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\Big(\int_0^s \frac{(s-u)^{\beta-1}}{\Gamma(\beta)}(h(u)-\lambda f(u,D^\delta x_n(u))\\&-g(u,x_n(u))du-\frac{k}{s^{\alpha-\beta}}x_n(s)\Big)ds-\frac{t^\alpha}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}\\&\times\Big(\int_0^s \frac{(s-u)^{\beta-1}}{\Gamma(\beta)}(h(u)-\lambda f(u,D^\delta x_n(u))-g(u,x_n(u)))du\\&-\frac{k}{s^{\alpha-\beta}}x_n(s)\Big)ds-\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\Big(\int_0^s\frac{(s-u)^{\beta-1}}{\Gamma(\beta)}(h(u)\\&-\lambda f(u,D^\delta x(u))-g(u,x(u)))du-\frac{k}{s^{\alpha-\beta}}x(s)\Big)ds-\frac{t^\alpha}{\Gamma(\alpha)}\\&\times\int_0^1(1-s)^{\alpha-1}\Big(\int_0^s\frac{(s-u)^{\beta-1}}{\Gamma(\beta)}(h(u)-\lambda f(u,D^\delta x(u))\\&-g(u,x(u))du-\frac{k}{s^{\alpha-\beta}}x(s)\Big)ds\Big|.\end{aligned} \tag{3.13}$$

Hence,

$$\begin{aligned}\|\mathcal{H}x_n-\mathcal{H}x\|_\infty \leq\ & 2|\lambda|\ \|f(.,D^\delta x_n(.))-f(.,D^\delta x(.))\|\ \frac{\beta(\beta+1,\alpha)}{\Gamma(\alpha)\Gamma(\beta+1)}\\&+2\|g(.,x_n(.))-g(.,x(.))\|\ \frac{\beta(\beta+1,\alpha)}{\Gamma(\alpha)\Gamma(\beta+1)}\\&+2K\frac{\Gamma(\beta-\alpha+1)}{\Gamma(\beta+1)}\|x_n-x\|.\end{aligned} \tag{3.14}$$

By $(H1)$, we have:

$$\|\mathcal{H}x_n-\mathcal{H}x\|_\infty \longrightarrow 0 \text{ as } n\longrightarrow\infty \tag{3.15}$$

Similarly, it can be shown that

$$\begin{aligned}\|D^{\delta}\mathcal{H}x_n - D^{\delta}\mathcal{H}x\|_{\infty} &\leq \left(\frac{|\lambda|}{\Gamma(\alpha+\beta-\delta+1)} + \frac{\beta\ (\beta+1,\alpha)|\lambda|}{\Gamma(\alpha-\delta+1)}\right)\\ &\times\|f(.,D^{\delta}x_n(.)) - f(.,D^{\delta}x(.))\|\\ &+\left(\frac{1}{\Gamma(\alpha+\beta-\delta+1)} + \frac{\beta\ (\beta+1,\alpha)}{\Gamma(\alpha-\delta+1)}\right)\\ &\times\|g(.,x_n(.)) - g(.,x(.))\|\\ &+\left(K\,\frac{\Gamma(\beta-\alpha+1)}{\Gamma(\beta-\delta+1)} + k\frac{\alpha\beta\ (\beta-\alpha+1,\alpha)}{\Gamma(\alpha-\delta+1)}\right)\\ &\times\|x_n - x\|.\end{aligned} \tag{3.16}$$

Then,

$$\|D^{\delta}\mathcal{H}x_n - D^{\delta}\mathcal{H}x\|_{\infty} \longrightarrow 0 \text{ as } n \longrightarrow \infty \tag{3.17}$$

From (3.15) and (3.17), we conclude that

$$\|\mathcal{H}x_n - \mathcal{H}x\|_X \longrightarrow 0\ as\ \ n \longrightarrow \infty$$

Consequently, $\mathcal{H}$ is continuous on X

Step 2: $\mathcal{H}$ maps bounded sets into bounded sets in X.
Indeed, it is enough to show that for any $r > 0$, there exists a positive constant p such that for each $y \in B_r = \{y \in X\ ; ||y||_X \leq r\}$ one has $||\mathcal{H}y||_X \leq p$. Let $x \in B_r$. We put:

$$p \ := \ Max\begin{cases} 2M_{h,f,g}\dfrac{\beta(\beta+1,\alpha)}{\Gamma(\alpha)\Gamma(\beta+1)} + 2rK\dfrac{\Gamma(\beta-\alpha+1)}{\Gamma(\beta+1)};\\ M_{h,f,g}\left(\dfrac{1}{\Gamma(\alpha+\beta-\delta+1)} + \dfrac{\beta\ (\beta+1,\alpha)}{\Gamma(\alpha-\delta+1)}\right)\\ +Kr\left(\dfrac{\Gamma(\beta-\alpha+1)}{\Gamma(\beta-\delta+1)} + \dfrac{\alpha\ \beta\ (\beta-\alpha+1,\alpha)}{\Gamma(\alpha-\delta+1)}\right).\end{cases}$$

Using $(H3)$and $(H4)$, we can write

$$\|\mathcal{H}x\| \leq 2M_{h,f,g}\frac{\beta(\beta+1,\alpha)}{\Gamma(\alpha)\Gamma(\beta+1)} + 2rK\frac{\Gamma(\beta-\alpha+1)}{\Gamma(\beta+1)}. \tag{3.18}$$

Moreover, we have:

$$\begin{aligned}\|D^{\delta}\mathcal{H}x\| \quad &\leq \quad M_{h,f,g}\left(\frac{1}{\Gamma(\alpha+\beta-\delta+1)}+\frac{\beta\ (\beta+1,\alpha)}{\Gamma(\alpha-\delta+1)}\right)\\ &\quad +Kr\Big(\frac{\Gamma(\beta-\alpha+1)}{\Gamma(\beta-\delta+1)}+\frac{\alpha\ \beta\ (\beta-\alpha+1,\alpha)}{\Gamma(\alpha-\delta+1)}\Big).\end{aligned} \tag{3.19}$$

We deduce that

$$\|\mathcal{H}y\|_X \leq p.$$

Consequently, $\mathcal{H}$ is uniformly bounded on Br.

Step 3: $\mathcal{H}$ maps bounded sets into equicontinuous sets of X.

Let $t_1, t_2 \in [0,1], t_1 < t_2$ and let B_r be a bounded set of X as in Step 2.Let $x \in B_r$ Then for each $t \in J$ we have

$$\begin{aligned}|\mathcal{H}x(t_1)-\mathcal{H}x(t_2)| = &|\frac{1}{\Gamma(\alpha)}\int_0^{t_1}(t_1-s)^{\alpha-1}\Big(\int_0^s\frac{(s-u)^{\beta-1}}{\Gamma(\beta)}(h(u)-\lambda f(u,D^{\delta}x(u))\\ &-g(u,x(u))du-\frac{k}{s^{\alpha-\beta}}x(s)\Big)ds-\frac{{t_1}^{\alpha}}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}\\ &\times\Big(\int_0^s\frac{(s-u)^{\beta-1}}{\Gamma(\beta)}(h(u)-\lambda f(u,D^{\delta}x(u))-g(u,x(u)))du\\ &-\frac{k}{s^{\alpha-\beta}}x(s)\Big)ds-{t_1}^{\alpha}(m-m^*)-\frac{1}{\Gamma(\alpha)}\int_0^{t_2}(t_2-s)^{\alpha-1}\\ &\times\Big(\int_0^s\frac{(s-u)^{\beta-1}}{\Gamma(\beta)}(h(u)-\lambda f(u,D^{\delta}y(u))-g(u,y(u)))du\\ &-\frac{k}{s^{\alpha-\beta}}y(s)\Big)ds+\frac{{t_2}^{\alpha}}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}\Big(\int_0^s\frac{(s-u)^{\beta-1}}{\Gamma(\beta)}(h(u)\\ &-\lambda f(u,D^{\delta}y(u))-g(u,y(u))du-\frac{k}{s^{\alpha-\beta}}y(s)\Big)ds+{t_2}^{\alpha}(m-m^*)|.\end{aligned} \tag{3.20}$$

Then,

$$
\begin{aligned}
|\mathcal{H}x(t_1)-\mathcal{H}x(t_2)| &\leq \frac{M_{h,f,g}}{\Gamma(\beta+\alpha+1)}\Big(({t_1}^{\beta+\alpha}-{t_2}^{\beta+\alpha})+({t_1}^{\alpha}-{t_2}^{\alpha})\Big)\\
&+kr\frac{\Gamma(\beta-\alpha+1)}{\Gamma(\beta+1)}\Big(({t_2}^{\beta}-{t_1}^{\beta})+({t_1}^{\alpha}-{t_2}^{\alpha})\Big)\\
&+|m-m^*|({t_1}^{\alpha}-{t_2}^{\alpha}),
\end{aligned}
\tag{3.21}
$$

and

$$
\begin{aligned}
|D^{\delta}\mathcal{H}x(t_1)-D^{\delta}\mathcal{H}x(t_2)| &\leq \frac{M_{f,g,h}}{\Gamma(\beta+\alpha-\delta+1)}({t_1}^{\beta+\alpha-\delta}-{t_2}^{\beta+\alpha-\delta})\\
&+rk\frac{\Gamma(\beta-\alpha+1)}{\Gamma(\beta-\delta+1)}({t_1}^{\beta-\delta}-{t_2}^{\beta-\delta})\\
&+\frac{M_{f,g,h}\,\Gamma(\alpha+1)}{\Gamma(\alpha-\delta+1)\Gamma(\alpha+\beta+1)}(\,{t_1}^{\alpha-\delta}-{t_2}^{\alpha-\delta})\\
&+kr\frac{\Gamma(\alpha+1)\Gamma(\beta-\alpha+1)}{\Gamma(\alpha-\delta+1)\Gamma(\beta+1)}({t_1}^{\alpha-\delta}-{t_2}^{\alpha-\delta})\\
&+\frac{\Gamma(\alpha+1)|m-m^*|}{\Gamma(\alpha-\delta+1)}({t_1}^{\alpha-\delta}-{t_2}^{\alpha-\delta}).
\end{aligned}
\tag{3.22}
$$

The right hand sides of (3.21) and (3.22) tend to zero independently of $x \in Br$ as $t_1 \to t_2$. As a consequence of Steps 1,2,3 together with the AscoliâĂŞArzela theorem, we can conclude that $\mathcal{H}$ is completely continuous.

Step 4: The set $A=\{x \in X : x=\sigma\mathcal{H}x, \sigma \in]0,1[\}$ is bounded.
Let $y \in A,\ \ t \in J$. Then, we have $y=\sigma\mathcal{H}y, 0<\sigma<1$. Hence, we can write

$$
\|y\|_\infty \leq 2\Big(|\lambda|\ M_f+M_g\Big)\ \frac{\beta(\beta+1,\alpha)}{\Gamma(\alpha)\Gamma(\beta+1)}+2rK\frac{\Gamma(\beta-\alpha+1)}{\Gamma(\beta+1)}. \tag{3.23}
$$

$$
\begin{aligned}
\|D^{\delta}y\|_\infty &\leq (M_{f,g,h})\left(\frac{1}{\Gamma(\alpha+\beta-\delta+1)}+\frac{\beta\ (\beta+1,\alpha)}{\Gamma(\alpha-\delta+1)}\right)\\
&+Kr\Big(\frac{\Gamma(\beta-\alpha+1)}{\Gamma(\beta-\delta+1)}+\frac{\alpha\ \beta\ (\beta-\alpha+1,\alpha)}{\Gamma(\alpha-\delta+1)}\Big).
\end{aligned}
\tag{3.24}
$$

Therefore

$$\|y\|_X \leq Max \begin{cases} 2\Big(|\lambda|\ M_f + M_g\Big)\ \dfrac{\beta(\beta+1,\alpha)}{\Gamma(\alpha)\Gamma(\beta+1)} + 2rK\dfrac{\Gamma(\beta-\alpha+1)}{\Gamma(\beta+1)}, \\ M_{f,g,h}\ \left(\dfrac{1}{\Gamma(\alpha+\beta-\delta+1)} + \dfrac{\beta\ (\beta+1,\alpha)}{\Gamma(\alpha-\delta+1)}\right) \\ +Kr\Big(\dfrac{\Gamma(\beta-\alpha+1)}{\Gamma(\beta-\delta+1)} + \dfrac{\alpha\ \beta\ (\beta-\alpha+1,\alpha)}{\Gamma(\alpha-\delta+1)}\Big). \end{cases}$$

Consequently, $\|y\|_X \leq \sigma p\ < \infty$, the set is thus bounded.
As a consequence of Schaefer fxed point theorem, we deduce that $\mathcal{H}$ has a fixed point which is a solution of the problem (1).

□

3.3. **UH-Stability.**

Definition 3.1. The equation (1.1) has the UH stability if there exists a real number $R > 0$, such that for each $\varepsilon > 0$, for any $t \in J$, and for each $x \in X$ solution of the inequality

$$|D^{\beta}(D^{\alpha} + \frac{k}{t^{\alpha-\beta}})x(t) + \lambda f(t, D^{\delta}x(t)) + g(t, x(t)) - h(t)| \quad < \quad \varepsilon, \tag{3.25}$$

there exists a solution $y \in X$ of (1.1); that is

$$D^{\beta}(D^{\alpha} + \frac{k}{t^{\alpha-\beta}})y(t) + \lambda f(t, D^{\delta}y(t)) + g(t, y(t)) \quad = \quad h(t), \text{Ât}, \tag{3.26}$$

such that,

$$\|x - y\|_X \leq R\varepsilon.$$

Definition 3.2. The equation (1.1) has the UH stability in the generalized sense if there exists $\varphi \in C(\mathbb{R}^+, \mathbb{R}^+)$, such that $\varphi(0) = 0$: for each $\varepsilon > 0$, and for any $x \in X$ solution of

$$|D^{\beta}(D^{\alpha} + \frac{k}{t^{\alpha-\beta}})x(t) + \lambda f(t, D^{\delta}x(t)) + g(t, x(t)) - h(t)| < \varepsilon, \tag{3.27}$$

there exists a solution $y \in X$ of equation (1.1), such that

$$\|x - y\|_X < \varphi(\varepsilon).$$

Theorem 3.3. *Let the assumptions of Theorem (3.1) hold and* $|\lambda|L_f + L_g < 1$*. If the inequality*

$$\|D^{\beta}(D^{\alpha}+\tfrac{k}{t^{\alpha-\beta}})x\| \geq Max \begin{cases} 2\frac{M_{h,f,g}+\left(|\lambda|L_f+L_g\right)r}{\Gamma(\alpha+\beta+1)}+r\frac{k\Gamma(\beta-\alpha+1)}{\Gamma(\beta+1)}+2|m|+|m^*|; \\ \Big(M_{h,f,g}+(|\lambda|L_f+L_g)r\Big)\Big(\frac{1}{\Gamma(\beta+\alpha-\delta+1)} \\ +\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+\beta+1)\Gamma(\alpha-\delta+1)}\Big) \\ +kr\Big(\dfrac{\Gamma(\alpha+1)\Gamma(\beta+\alpha+1)}{\Gamma(\alpha-\delta+1)\Gamma(\beta+1)}+\dfrac{\Gamma(\beta-\alpha+1)}{\Gamma(\beta-\delta+1)}\Big) \\ +\frac{\Gamma(\alpha+1)}{\Gamma(\alpha-\delta+1)}|m-m^*| \end{cases} \tag{3.28}$$

is valid, then problem(1.1) has the UH stability.

Proof. Let ε and let $x \in X$ be a function which satisfies (3.25) and let $y \in X$ be the unique solution of the equation (1.1). We have:

$$\|x\|_\infty \leq Max \begin{cases} 2\dfrac{M_{h,f,g}+(|\lambda|L_f+L_g)r}{\Gamma(\alpha+\beta+1)}+r\dfrac{k\Gamma(\beta-\alpha+1)}{\Gamma(\beta+1)}+2|m|+|m^*|; \\ \Big(M_{h,f,g}+(|\lambda|L_f+L_g)r\Big)\Big(\dfrac{1}{\Gamma(\beta+\alpha-\delta+1)} \\ +\dfrac{\Gamma(\alpha+1)}{\Gamma(\alpha+\beta+1)\Gamma(\alpha-\delta+1)}\Big) \\ +kr\Big(\dfrac{\Gamma(\alpha+1)\Gamma(\beta+\alpha+1)}{\Gamma(\alpha-\delta+1)\Gamma(\beta+1)}+\dfrac{\Gamma(\beta-\alpha+1)}{\Gamma(\beta-\delta+1)}\Big) \\ +\dfrac{\Gamma(\alpha+1)}{\Gamma(\alpha-\delta+1)}|m-m^*| \end{cases} \tag{3.29}$$

Combining (3.28) and (3.29), we obtain

$$\|x\|_\infty \quad \leq \quad \|D^{\beta}(D^{\alpha}+\frac{k}{t^{\alpha-\beta}})x\| \tag{3.30}$$

Therefore, we get

$$\begin{aligned}\|x-y\|_\infty &\le \|D^\beta(D^\alpha + \frac{k}{t^{\alpha-\beta}}(x-y)\| \\ &\le \sup_{t\in J}|D^\beta(D^\alpha + \frac{k}{t^{\alpha-\beta}})x(t) - D^\beta(D^\alpha + \frac{k}{t^{\alpha-\beta}})y(t) \\ &+\lambda f(t, D^\delta x(t)) + g(t,x(t)) - h(t) - \lambda f(t, D^\delta y(t)) - g(t,y(t)) - h(t) \\ &-\lambda f(t, D^\delta x(t)) - g(t,x(t)) + h(t) + \lambda f(t, D^\delta y(t)) + g(t,y(t)) + h(t)|\end{aligned} \tag{3.31}$$

Thanks to (3.25) and (3.26), we get

$$\|x-y\|_\infty \le \varepsilon + \lambda|L_f||D^\delta x - D^\delta y|| + L_g||x-y||. \tag{3.32}$$

But since,

$$|\lambda|L_f + L_g < 1,$$

then, we can write

$$||x-y||_\infty \le \frac{\varepsilon}{1-(|\lambda|L_f + L_g)} = \varepsilon R. \tag{3.33}$$

On the other hand,

$$\|D^\delta x\|_\infty \le \|D^\beta(D^\alpha + \frac{k}{t^{\alpha-\beta}})x\|. \tag{3.34}$$

So,

$$||D^\delta(x-y)||_\infty \le \frac{\varepsilon}{1-(|\lambda|L_f + L_g)} = \varepsilon R. \tag{3.35}$$

By (3.33) and (3.35), we get

$$||x-y||_X \le \frac{\varepsilon}{1-(|\lambda|L_f + L_g)} = \varepsilon R \tag{3.36}$$

Consequently, (1.1) has the UH stability.
Taking $\varphi(\varepsilon) = \varepsilon R$, we can state that the equation (1.1) has the generalized UH stability.

□

4. Conclusion

We have studied a singular differential problem of Lane Emden type. By considering a sequential equation that involves a Caputo operator of type $D^\beta(D^\alpha))$ that depends on two parameters of derivation without commutativity properties and by taking a δ Caputo derivative in the right hand side of the equation, we have proved an existence and uniqueness theorem for our problem. Then, based on Schaefer fixed point theorem, an existence result for our problem has been studied. This second result establishes some sufficient conditions assuring at least one solution for the problem. At the end, two UH stability definitions have been introduced and UH stability results have been delivered.

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