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ON SIMPLE JORDAN TYPE INEQUALITIES

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Abstract. In this paper, simple rational bounds for the functions of the type $\frac{f(x)}{x}$ or $\frac{x}{f(x)}$, where $f(x)$ is circular or hyperbolic function are obtained. The inequalities thus established are sufficiently sharp. In particular, some new lower and upper bounds of $\sin x/x$, $x/\sinh x$, $x/\tan x$ and $\tanh x/x$ are proposed. The proposed bounds are even functions.

1. Introduction

The two sided inequality [5]

$$\frac{2}{\pi} < \frac{\sin x}{x} < 1, \ x \in (0, \pi/2) \tag{1.1}$$

is due to Jordan. Many inequalities of this type have been proposed and refined by Mathematicians so far [2–7, 9, 10, 12–30]. We give a short summary of some already proved results relating the main results of this paper.

R. Klén, M. Visuri et al. [12] proved the following inequalities

$$\frac{6-x^2}{6} < \frac{\sin x}{x} < 1 - \frac{2x^2}{3\pi^2}; \ x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \tag{1.2}$$

$$\frac{\sin x}{x} < \frac{x}{\sinh x}; \ x \in \left(0, \frac{\pi}{2}\right), \tag{1.3}$$

$$\cosh x < \frac{1}{\cos x}; \ x \in (0, 1), \tag{1.4}$$

and

$$\frac{5}{5+x^2} < \frac{x}{\sinh x} < \frac{6}{6+x^2}; \ x \in (0, 1). \tag{1.5}$$

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In fact (1.4) is true in $(0, \pi/2)$. The double inequality

$$\frac{\pi^2 - 4x^2}{\pi^2} < \frac{x}{\tan x} < \frac{\pi^2 - 4x^2}{8}; \; 0 < x < \frac{\pi}{2} \tag{1.6}$$

is due to M. Becker and E. L. Stark [10]. The bounds of $\frac{\tanh x}{x}$ in [13] for $x \in (0, 1)$ are given as below

$$e^{-x^2/3} < \frac{\tanh x}{x} < e^{-\lambda x^2}; \;\; \lambda \approx 0.272342. \tag{1.7}$$

In this work, we aim to refine lower bound of (1.5), upper bounds of (1.2), (1.6) and give more sharp bounds of (1.7) by using simple rational functions. Though there are stronger bounds in the recent papers([6, 11–16, 21–30]) than the bounds obtained in this paper, they are much complex. Our bounds are easy to deal with.

2. Main Results and Their Proofs

To obtain our main results we shall use L'Hospital's Rule of Monotonicity [8] which is given below.

Lemma 2.1. *(The L'Hospital's rule [8] in monotone form) : Let $f, g : [a, b] \to \mathbb{R}$ be two continuous functions which are differentiable on (a, b) and $g' \neq 0$ in (a, b). If f'/g' is increasing (or decreasing) on (a, b), then the functions $\frac{f(x)-f(a)}{g(x)-g(a)}$ and $\frac{f(x)-f(b)}{g(x)-g(b)}$ are also increasing (or decreasing) on (a, b). If f'/g' is strictly monotone, then the monotonicity in the conclusion is also strict.*

Lemma 2.2. *For $x \in (0, \infty)$*

$$\frac{x}{\sinh x} < \cosh x \tag{2.1}$$

holds.

Proof. We know that $x < \sinh x$ and $\cosh x > 1$ in $(0, \infty)$. This proves the lemma.

We now state our Main results and their proofs as follows.

Theorem 2.1. *For $x \in (0, \pi/2)$, one has*

$$\frac{a}{a + x^2} < \frac{\sin x}{x} < \frac{b}{b + x^2} \tag{2.2}$$

with the best possible constants $a \approx 4.322735$ and $b = 6$.
Equivalently,

$$\frac{\pi^2}{\pi^2 + 2(\pi - 2)x^2} < \frac{\sin x}{x} < \frac{6}{6 + x^2}.$$

Proof. Let

$$f(x) = \frac{x^2 \sin x}{x - \sin x} = \frac{f_1(x)}{f_2(x)},$$

where $f_1(x) = x^2 \sin x$ and $f_2(x) = x - \sin x$ with $f_1(0) = f_2(0) = 0$. Differentiation gives

$$\frac{f_1'(x)}{f_2'(x)} = \frac{x^2 \cos x + 2x \sin x}{1 - \cos x} = \frac{f_3(x)}{f_4(x)}$$

where $f_3(x) = x^2 \cos x + 2x \sin x$ and $f_4(x) = 1 - \cos x$ with $f_3(0) = f_4(0) = 0$. By differentiation we get

$$\frac{f_3'(x)}{f_4'(x)} = 4x \cot x - x^2 + 2 = g(x).$$

Now,

$$g'(x) = 4[\cot x - x \operatorname{cosec}^2 x] - 2x.$$

Using (1.3), (1.4) and (2.1), we have

$$\cos x < \frac{x}{\sin x} \; i.e. \frac{\cos x}{\sin x} < \frac{x}{\sin^2 x}; x \in (0, \pi/2),$$

which implies that,

$$\cot x < \frac{x}{\sin^2 x} + \frac{x}{2}.$$

Therefore, $4[\cot x - x\operatorname{cosec}^2 x] - 2x < 0$. i.e. $g'(x) < 0$. By Lemma 2.1, $f(x)$ is strictly decreasing in $(0, \pi/2)$. So that

$$f(0+) > f(x) > f(\pi/2) \text{ ; for any } x \text{ in } (0, \pi/2).$$

Consequently, $a = f(\pi/2) = \frac{(\pi/2)^2}{\pi/2-1} \approx 4.322735$ and $b = f(0+) = 6$, by L'Hospital's rule. ☐

Note: For more sharp bounds of $\frac{\sin x}{x}$ one may refer the recent papers e.g. [1],[2],[9],[11], [21], [23], [29] etc.

Remark 2.1. Combining (1.2) and (2.2), we have

$$\frac{6 - x^2}{6} < \frac{\sin x}{x} < \frac{6}{6 + x^2}; x \in (0, \pi/2). \tag{2.3}$$

The upper bound of (1.2) is sharpened in Theorem 2.1. Next we refine the lower bound of (1.5).

Theorem 2.2. *For $x \in (0, 1)$, the inequalities*

$$\frac{c}{c + x^2} < \frac{x}{\sinh x} < \frac{6}{6 + x^2} \tag{2.4}$$

are true with the best possible constants $c \approx 5.707724$ and 6.

Proof. Introduce

$$f(x) = \frac{x^3}{\sinh x - x} = \frac{f_1(x)}{f_2(x)}$$

where $f_1(x) = x^3$ and $f_2(x) = \sinh x - x$ with $f_1(0) = 0 = f_2(0)$. Differentiation gives us

$$\frac{f_1'(x)}{f_2'(x)} = \frac{3x^2}{\cosh x - 1} = \frac{f_3(x)}{f_4(x)}$$

where $f_3(x) = 3x^2$ and $f_4(x) = \cosh x - 1$ with $f_3(0) = f_4(0) = 0$. By differentiation we get

$$\frac{f_3'(x)}{f_4'(x)} = \frac{6x}{\sinh x} = \frac{f_5(x)}{f_6(x)}$$

where $f_5(x) = 6x$ and $f_6(x) = \sinh x$ with $f_5(0) = f_6(0) = 0$. Again by differentiation

$$\frac{f_5'(x)}{f_6'(x)} = \frac{6}{\cosh x},$$

which is clearly strictly decreasing in $(0,1)$. By Lemma 2.1, $f(x)$ is strictly decreasing in $(0,1)$. On account of which,

$$f(0+) > f(x) > f(1) \text{ for } 0 < x < 1.$$

Therefore, $c = f(1) = \frac{1}{\sinh 1 - 1} \approx 5.707724$ and $f(0+) = 6$, by L'Hospital's rule. □

Remark 2.2. The statement of Theorem 2.2 can be generalized for $x \in (0,\eta)$ where $\eta > 0$ with $c = \frac{\eta^3}{\sinh \eta - \eta}$.

Now we sharp the upper bound of (1.6) by using rational function. More sharp and generalized versions are given by Marija Nenezić and Ling Zhu in [11].

Theorem 2.3. *For $x \in (0,\delta)$ where $\delta < \frac{\pi}{2}$ one has*

$$\frac{k}{k+x^2} < \frac{x}{\tan x} < \frac{3}{3+x^2} \tag{2.5}$$

with the best possible constants $k = \frac{\delta^3}{\tan\delta - \delta}$ and 3.

Proof. Let

$$f(x) = \frac{x^3}{\tan x - x} = \frac{f_1(x)}{f_2(x)}$$

where $f_1(x) = x^3$ and $f_2(x) = \tan x - x$ with $f_1(0) = f_2(0) = 0$. By differentiation we have

$$\frac{f_1'(x)}{f_2'(x)} = \frac{3x^2}{\sec^2 x - 1} = \frac{f_3(x)}{f_4(x)}$$

where $f_3(x) = 3x^2$ and $f_4(x) = \sec^2 x - 1$ with $f_3(0) = f_4(0) = 0$. Again differentiating

$$\frac{f_3'(x)}{f_4'(x)} = \frac{3x}{\sec^2 x \tan x} = \frac{f_5(x)}{f_6(x)}$$

where $f_5(x) = 3x$ and $f_6(x) = \sec^2 x \tan x$ with $f_5(0) = f_6(0) = 0$. Differentiation gives us

$$\frac{f_5'(x)}{f_6'(x)} = \frac{3}{\sec^4 x + 2\sec^2 x \tan^2 x}$$

which is clearly decreasing in $(0,\pi/2)$. By Lemma 2.1, $f(x)$ is also decreasing in $(0,\pi/2)$. Therefore,

$$f(0+) > f(x) > f(\pi/2) \text{ ; for any } x \text{ in } (0,\pi/2).$$

Hence, $k = f(\delta) = \frac{\delta^3}{\tan\delta - \delta}$ and $f(0+) = 3$, by L'Hospital's rule. □

Lastly, we give sharp bounds of (1.7).

Theorem 2.4. *For $x \in (0,1)$, one has*

$$\frac{3}{3+x^2} < \frac{\tanh x}{x} < \frac{d}{d+x^2} \tag{2.6}$$

where the constants 3 and $d \approx 3.194528$ are best possible.

Proof. Introduce
$$f(x) = \frac{x^2 \tanh x}{x - \tanh x} = \frac{f_1(x)}{f_2(x)}$$
where $f_1(x) = x^2 \tanh x$ and $f_2(x) = x - \tanh x$ with $f_1(0) = f_2(0) = 0$. Differentiation gives
$$\frac{f_1'(x)}{f_2'(x)} = x^2 \text{cosech}^2 x + 2x \coth x = h(x).$$
Hence,
$$h'(x) = 2 \coth x(1 - x^2 \text{cosech}^2 x).$$
Now, $\coth x$ is positive in $(0, 1)$ and $1 - x^2 \text{cosech}^2 x > 0$, as $x < \sinh x$. This yields, $h'(x) > 0$. Therefore $h(x)$ is strictly increasing in $(0, 1)$. By Lemma 2.1, $f(x)$ is strictly increasing in $(0, 1)$, so
$$f(0+) < f(x) < f(1) \text{ ; for } x \in (0, \pi/2).$$
Consequently, $f(0+) = 3$ by l'Hôpital's rule and $d = f(1) = \frac{\tanh 1}{1 - \tanh 1} \approx 3.194528$. □

Remark 2.3. The statement of Theorem 2.4 can be generalized for $x \in (0, \mu)$ where $\mu > 0$ with $d = \frac{\mu^2 \tanh \mu}{\mu - \tanh \mu}$.

Note: For even more stronger upper bound of (2.6), one may refer [3].

Remark 2.4. As a corollary, Theorem 2.3 and Theorem 2.4 give us
$$\frac{x}{\tan x} < \frac{\tanh x}{x}; x \in (0, 1) \tag{2.7}$$
which has been already proved in [13].

Corollary 2.1. *For $x \in (0, 1)$*
$$\sin x \sinh x < \tan x \tanh x \tag{2.8}$$

Proof. The proof follows immediately by combining (1.3) and (2.7). □

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