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REFINEMENTS OF JENSEN'S INEQUALITY FOR INFINITE CONVEX COMBINATIONS

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Abstract. Jensen's inequality for infinite convex combinations is studied in order to obtain its refinements. The article is relied on two basic double inequalities related to convex functions defined on the bounded closed interval of real numbers. Convex combinations with the same center are employed.

1. Introduction

A bounded closed interval of real numbers is the basis for studying convex analysis. Let a and b be real points such that $a < b$. The closed interval or segment between points a and b can be appointed as the set

$$[a, b] = \{\alpha a + \beta b : \alpha, \beta \in [0, 1],\ \alpha + \beta = 1\},$$

that is, as the set of all binomial convex combinations of points a and b. In this regard, each point $x \in [a, b]$ can be represented as the binomial convex combination

$$x = \frac{b - x}{b - a}a + \frac{x - a}{b - a}b,$$

which shows that $\alpha(x) = (b - x)/(b - a)$ and $\beta(x) = (x - a)/(b - a)$.

An n-membered linear combination $\sum_{i=1}^{n} \lambda_i x_i$ of points $x_i \in [a, b]$ is said to be convex if coefficients $\lambda_i \in [0, 1]$ and $\sum_{i=1}^{n} \lambda_i = 1$. Using the binomial convex combinations $x_i = \alpha_i a + \beta_i b$, it follows that

$$\sum_{i=1}^{n} \lambda_i x_i = \Big(\sum_{i=1}^{n} \lambda_i \alpha_i\Big)a + \Big(\sum_{i=1}^{n} \lambda_i \beta_i\Big)b.$$

Taking $\alpha = \sum_{i=1}^{n} \lambda_i \alpha_i$ and $\beta = \sum_{i=1}^{n} \lambda_i \beta_i$, we have $\alpha, \beta \in [0, 1]$ and $\alpha + \beta = 1$, which shows that the point $c = \sum_{i=1}^{n} \lambda_i x_i = \alpha a + \beta b$ belongs to $[a, b]$. The point c itself is usually called a combination center.

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2. Two basic inequalities with convex combinations

In creating mathematical inequalities, the main mediator is a convex function. A function $f : [a, b] \to \mathbb{R}$ is said to be convex if the inequality

$$f(\alpha x + \beta y) \leq \alpha f(x) + \beta f(y)$$

holds for every binomial convex combination $\alpha x + \beta y$ of points $x, y \in [a, b]$.

Let $\sum_{i=1}^{n} \lambda_i x_i$ be a convex combination of points $x_i \in [a, b]$. If $\alpha a + \beta b$ is the convex combination of endpoints a and b such that

$$\alpha a + \beta b = \sum_{i=1}^{n} \lambda_i x_i,$$

then each convex function $f : [a, b] \to \mathbb{R}$ satisfies the double inequality

$$f(\alpha a + \beta b) \leq \sum_{i=1}^{n} \lambda_i f(x_i) \leq \alpha f(a) + \beta f(b). \tag{2.1}$$

This fundamental inequality says that the convex function values, taken in the forms of convex combinations, grow from the center to the ends. The left-hand side (containing the first and second members) of the above inequality written as

$$f\left(\sum_{i=1}^{n} \lambda_i x_i\right) \leq \sum_{i=1}^{n} \lambda_i f(x_i) \tag{2.2}$$

represents the discrete form of Jensen's inequality (see [1]), and it can be proved by mathematical induction. The right-hand side (containing the second and third members) of the inequality in formula (2.1) can be proved by multiplying each inequality $f(x_i) = f(\alpha_i a + \beta_i b) \leq \alpha_i f(a) + \beta_i f(b)$ with λ_i, and then summing up over $i = 1, \ldots, n$. The double inequality in formula (2.1) can also be proved by using the support and secant lines, it was demonstrated in [7, Theorem 3.1].

Let $[a_0, b_0]$ be a closed subinterval of $[a, b]$, where $a_0 < b_0$. Let $\sum_{i=1}^{n} \lambda_i x_i$ be a convex combination of points $x_i \in [a_0, b_0]$, and let $\sum_{j=1}^{m} \kappa_j y_j$ be a convex combination of points $y_j \in [a, b] \setminus (a_0, b_0)$. If the above combinations have the same center c taken as the convex combination $c = \alpha_0 a_0 + \beta_0 b_0$, that is, if

$$\sum_{i=1}^{n} \lambda_i x_i = \alpha_0 a_0 + \beta_0 b_0 = \sum_{j=1}^{m} \kappa_j y_j,$$

then each convex function $f : [a, b] \to \mathbb{R}$ satisfies the double inequality

$$\sum_{i=1}^{n} \lambda_i f(x_i) \leq \alpha_0 f(a_0) + \beta_0 f(b_0) \leq \sum_{j=1}^{m} \kappa_j f(y_j). \tag{2.3}$$

The above inequality, even more clearly then the inequality in formula (2.1), shows the nature of growth of the convex function values. The proof of this double inequality can be found in [5, Corollary 3.2].

More details on the convexity in general, convex functions and their inequalities can be found in books [9] and [10]. An extension of Jensen's inequality to affine combinations was obtained in [6]. New generalizations and refinements of Jensen's inequality were considered in [4], [3] and [2].

3. Inequalities with infinite convex combinations

In this section, we briefly recall some main results obtained in [8].

Definition 3.1. An infinite linear combination $\sum_{i=1}^{\infty}\lambda_i x_i$ of a real vector space points x_i is said to be convex if coefficients $\lambda_i \in [0,1]$ and their sum $\sum_{i=1}^{\infty}\lambda_i$ converges to number 1 in the field $\mathbb{R}$, usually written as $\sum_{i=1}^{\infty}\lambda_i = 1$.

A finite convex combination $\sum_{i=1}^{n}\lambda_i x_i$ can be taken as $\sum_{i=1}^{\infty}\lambda_i x_i$ with $\lambda_i = 0$ for every $i > n$.

Infinite convex combinations are prone to convergence.

Theorem A. *An infinite convex combination $\sum_{i=1}^{\infty}\lambda_i x_i$ of points $x_i \in [a,b]$ converges in $[a,b]$.*

Each infinite combination $\sum_{i=1}^{\infty}\lambda_i x_i$ with nonnegative coefficients λ_i such that $\sum_{i=1}^{\infty}\lambda_i = \lambda \in \mathbb{R}$ converges. Namely, the combination $\sum_{i=1}^{\infty}(\lambda_i/\lambda)x_i$ is convex.

Theorem A also refers to infinite convex combinations of the function values.

Corollary B. *Let X be a nonempty set, and let $g : X \to \mathbb{R}$ be a function with the image in $[a,b]$. Then an infinite convex combination $\sum_{i=1}^{\infty}\lambda_i g(x_i)$ of function values $g(x_i)$ with arguments $x_i \in X$ converges in $[a,b]$.*

The inequality in formula (2.1) can be expanded to infinite convex combinations.

Theorem C. *Let $\sum_{i=1}^{\infty}\lambda_i x_i$ be an infinite convex combination of points $x_i \in [a,b]$, and let $\alpha a + \beta b$ be the convex combination such that $\alpha a + \beta b = \sum_{i=1}^{\infty}\lambda_i x_i$.*

Then each convex function $f : [a,b] \to \mathbb{R}$ satisfies the double inequality

$$f(\alpha a + \beta b) \le \sum_{i=1}^{\infty}\lambda_i f(x_i) \le \alpha f(a) + \beta f(b). \tag{3.1}$$

The left-hand side (containing the first end second member) of the above inequality represented as

$$f\left(\sum_{i=1}^{\infty}\lambda_i x_i\right) \le \sum_{i=1}^{\infty}\lambda_i f(x_i) \tag{3.2}$$

is Jensen's inequality for infinite convex combinations.

The inequality in formula (2.3) can also be exposed in the infinite form.

Corollary D. *Let $[a_0,b_0]$ be a subinterval of $[a,b]$ with $a_0 < b_0$, let $\sum_{i=1}^{\infty}\lambda_i x_i$ be an infinite convex combination of points $x_i \in [a_0,b_0]$, and let $\sum_{i=1}^{\infty}\kappa_i y_i$ be an infinite convex combination of points $y_i \in [a,b] \setminus (a_0,b_0)$.*

If the above infinite combinations have the same center expressed by the convex combination $\alpha_0 a_0 + \beta_0 b_0$, that is, if

$$\sum_{i=1}^{\infty} \lambda_i x_i = \alpha_0 a_0 + \beta_0 b_0 = \sum_{i=1}^{\infty} \kappa_i y_i,$$

then each convex function $f : [a, b] \to \mathbb{R}$ satisfies the double inequality

$$\sum_{i=1}^{\infty} \lambda_i f(x_i) \leq \alpha_0 f(a_0) + \beta_0 f(b_0) \leq \sum_{i=1}^{\infty} \kappa_i f(y_i). \tag{3.3}$$

4. Main results

In this section, we investigate refinements of the Jensen and related inequalities with infinite convex combinations.

Using two infinite and two binomial convex combinations such that all have the same center, we can obtain the extended inequality which encompasses the inequalities in Theorem C and Corollary D.

Corollary 4.1. *Let $[a_0, b_0]$ be a subinterval of $[a, b]$ with $a_0 < b_0$, let $\sum_{i=1}^{\infty} \lambda_i x_i$ be an infinite convex combination of points $x_i \in [a_0, b_0]$, and let $\sum_{i=1}^{\infty} \kappa_i y_i$ be an infinite convex combination of points $y_i \in [a, b] \setminus (a_0, b_0)$.*

If the above infinite combinations have the same center expressed by the convex combinations $\alpha a + \beta b$ and $\alpha_0 a_0 + \beta_0 b_0$, that is, if

$$\alpha a + \beta b = \sum_{i=1}^{\infty} \lambda_i x_i = \alpha_0 a_0 + \beta_0 b_0 = \sum_{i=1}^{\infty} \kappa_i y_i,$$

then each convex function $f : [a, b] \to \mathbb{R}$ satisfies the multiple inequality

$$\begin{aligned} f(\alpha a + \beta b) &\leq \sum_{i=1}^{\infty} \lambda_i f(x_i) \\ &\leq \alpha_0 f(a_0) + \beta_0 f(b_0) \\ &\leq \sum_{i=1}^{\infty} \kappa_i f(y_i) \\ &\leq \alpha f(a) + \beta f(b). \end{aligned} \tag{4.1}$$

Proof. Extending the left-hand side (containing the first and second members) of the inequality in formula (3.1) with the inequality in formula (3.3), we get the inequality of the first four members in formula (4.1). The inequality of the fourth and fifth members in formula (4.1) is in fact the inequality of the second and third members in formula (3.1) because $\alpha a + \beta b = \sum_{i=1}^{\infty} \kappa_i y_i$. □

Taking $a_0 = a$ and $b_0 = b$, formula (4.1) is shortened to formula (3.1). In this case, the only choice for y_i is a or b.

In the next lemma, we demonstrate the refining method using two interposed points represented as the binomial convex combinations.

Lemma 4.1. *Let $x \in [a,b]$ be a point, let $\alpha a + \beta b$ be the convex combination such that $\alpha a + \beta b = x$, and let $p \geq 0$ be a number.*

Then each convex function $f : [a,b] \to \mathbb{R}$ satisfies the multiple inequality

$$\begin{aligned} f(\alpha a + \beta b) &\leq \alpha f\left(\frac{pa+x}{p+1}\right) + \beta f\left(\frac{x+pb}{p+1}\right) \\ &\leq \frac{p\alpha f(a) + f(x) + p\beta f(b)}{p+1} \\ &\leq \alpha f(a) + \beta f(b). \end{aligned} \tag{4.2}$$

Proof. Between a and x, and between x and b, we interpose the points

$$x_a = \frac{pa+x}{p+1} \quad \text{and} \quad x_b = \frac{x+pb}{p+1}.$$

It follows that $\alpha a + \beta b = x = \alpha x_a + \beta x_b$. Applying the convexity of f to the right side of the above equality, we get

$$f(\alpha a + \beta b) = f(\alpha x_a + \beta x_b) \leq \alpha f(x_a) + \beta f(x_b). \tag{4.3}$$

Further, applying the convexity of f to convex combinations

$$x_a = \frac{p}{p+1}a + \frac{1}{p+1}x \quad \text{and} \quad x_b = \frac{1}{p+1}x + \frac{p}{p+1}b,$$

we obtain

$$\begin{aligned} \alpha f(x_a) + \beta f(x_b) &\leq \frac{\alpha p}{p+1}f(a) + \frac{\alpha}{p+1}f(x) + \frac{\beta}{p+1}f(x) + \frac{\beta p}{p+1}f(b) \\ &= \frac{p\alpha f(a) + f(x) + p\beta f(b)}{p+1}. \end{aligned} \tag{4.4}$$

Since $f(x) = f(\alpha a + \beta b) \leq \alpha f(a) + \beta f(b)$, it follows that

$$\begin{aligned} \frac{p\alpha f(a) + f(x) + p\beta f(b)}{p+1} &\leq \frac{p\alpha f(a) + \alpha f(a) + \beta f(b) + p\beta f(b)}{p+1} \\ &= \alpha f(a) + \beta f(b). \end{aligned} \tag{4.5}$$

Connecting the inequalities in formula (4.3), formula (4.4) and formula (4.5), we gain the inequality in formula (4.2). □

We may use $\alpha = (b-x)/(b-a)$ and $\beta = (x-a)/(b-a)$ in formula (4.2). If $x = a$ or $x = b$, then all four members in formula (4.2) are equal to $f(x)$. The figurative form of the inequality in formula (4.2) is presented in Figure 1.

Remark 4.1. The limits of the interposed part (containing the second and third members) in formula (4.2) as p approaches zero and infinity are as follows.

The second and third members approach $f(\alpha a + \beta b)$ as p approaches zero.

The third member approaches $\alpha f(a) + \beta f(b)$ as p approaches infinity.

If $x \in (a,b)$, then the second member approaches $\alpha f(a+) + \beta f(b-)$ as p approaches infinity. This is true because the monotonicity and boundedness of f on $(a, a+\varepsilon)$ and $(b-\varepsilon, b)$ for sufficiently small $\varepsilon > 0$ provide the limits

$$\lim_{p\to\infty} f\left(\frac{pa+x}{p+1}\right) = f(a+)$$

and

$$\lim_{p\to\infty} f\left(\frac{x+pb}{p+1}\right) = f(b-).$$

In the light of the above considerations, we recall that the observed convex function f satisfies the boundary inequalities $f(a+) \leq f(a)$ and $f(b-) \leq f(b)$.

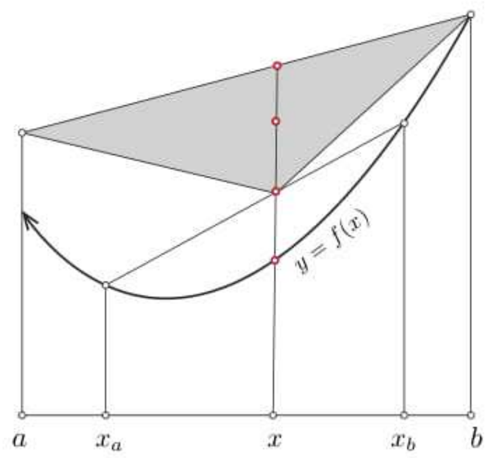

FIGURE 1. Visual presentation of the inequality in formula (4.2)

By combining infinite and binomial convex combinations, we can achieve the refinements of the fundamental inequality in formula (3.1).

Theorem 4.1. *Let $\sum_{i=1}^{\infty} \lambda_i x_i$ be an infinite convex combination of points $x_i \in [a,b]$, let $\alpha_i a + \beta_i b$ be convex combinations such that $\alpha_i a + \beta_i b = x_i$, let $\alpha a + \beta b$ be the convex combination such that $\alpha a + \beta b = \sum_{i=1}^{\infty} \lambda_i x_i$, and let $p \geq 0$ be a number.*

Then each convex function $f : [a,b] \to \mathbb{R}$ satisfies the multiple inequality

$$\begin{aligned} f(\alpha a + \beta b) &\leq \sum_{i=1}^{\infty} \lambda_i f(x_i) \\ &\leq \sum_{i=1}^{\infty} \lambda_i \left[\alpha_i f\left(\frac{pa+x_i}{p+1}\right) + \beta_i f\left(\frac{x_i+pb}{p+1}\right)\right] \\ &\leq \sum_{i=1}^{\infty} \lambda_i \frac{p\alpha_i f(a) + f(x_i) + p\beta_i f(b)}{p+1} \\ &\leq \alpha f(a) + \beta f(b). \end{aligned} \tag{4.6}$$

Proof. The inequality of the first and second members is verified in formula (3.1). The remaining inequality part (containing the second, third, fourth and fifth members) can be derived as follows.

Applying the inequality in formula (4.2) to the point x_i, we obtain

$$\begin{aligned} f(x_i) = f(\alpha_i a + \beta_i b) &\leq \alpha_i f\left(\frac{pa + x_i}{p+1}\right) + \beta_i f\left(\frac{x_i + pb}{p+1}\right) \\ &\leq \frac{p\alpha_i f(a) + f(x_i) + p\beta_i f(b)}{p+1} \\ &\leq \alpha_i f(a) + \beta_i f(b). \end{aligned}$$

Multiplying the above inequality with λ_i, and then summing up over $i = 1, \ldots, \infty$, we reach the remaining inequality part. The last member

$$\sum_{i=1}^{\infty} \lambda_i[\alpha_i f(a) + \beta_i f(b)] = \left(\sum_{i=1}^{\infty} \lambda_i \alpha_i\right) f(a) + \left(\sum_{i=1}^{\infty} \lambda_i \beta_i\right) f(b) = \alpha f(a) + \beta f(b)$$

because the equality

$$\alpha a + \beta b = \sum_{i=1}^{\infty} \lambda_i(\alpha_i a + \beta_i b) = \left(\sum_{i=1}^{\infty} \lambda_i \alpha_i\right) a + \left(\sum_{i=1}^{\infty} \lambda_i \beta_i\right) b$$

provides $\alpha = \sum_{i=1}^{\infty} \lambda_i \alpha_i$ and $\beta = \sum_{i=1}^{\infty} \lambda_i \beta_i$. Thus the above calculation completes the proof. $\square$

As regards the coefficients α_i and β_i in formula (4.6), since

$$x_i = \frac{b - x_i}{b - a} a + \frac{x_i - a}{b - a} b,$$

we may use

$$\alpha_i = \frac{b - x_i}{b - a} \quad \text{and} \quad \beta_i = \frac{x_i - a}{b - a}.$$

Some refinements of the inequality in formula (3.1) can be achieved by the skilful use of the convex combination center.

Theorem 4.2. *Let $\sum_{i=1}^{\infty} \lambda_i x_i$ be an infinite convex combination of points $x_i \in [a, b]$, let $c = \sum_{i=1}^{\infty} \lambda_i x_i$ be its center, and let $p \geq 0$ be a number.*

Then each convex function $f : [a, b] \to \mathbb{R}$ satisfies the multiple inequality

$$\begin{aligned} f\left(\sum_{i=1}^{\infty} \lambda_i x_i\right) &\leq \sum_{i=1}^{\infty} \lambda_i f\left(\frac{x_i + pc}{p+1}\right) \\ &\leq \sum_{i=1}^{\infty} \lambda_i \frac{f(x_i) + pf(c)}{p+1} \\ &\leq \sum_{i=1}^{\infty} \lambda_i f(x_i) \\ &\leq \frac{b - c}{b - a} f(a) + \frac{c - a}{b - a} f(b). \end{aligned} \tag{4.7}$$

Proof. Applying Jensen's inequality for infinite convex combinations in formula (3.2) to the right side of the convex combinations equality

$$\sum_{i=1}^{\infty}\lambda_i x_i = \frac{1}{p+1}\sum_{i=1}^{\infty}\lambda_i x_i + \frac{p}{p+1}c = \sum_{i=1}^{\infty}\lambda_i \frac{x_i+pc}{p+1},$$

we get

$$f\left(\sum_{i=1}^{\infty}\lambda_i x_i\right) \le \sum_{i=1}^{\infty}\lambda_i f\left(\frac{x_i+pc}{p+1}\right). \tag{4.8}$$

Further, applying the convexity of f to each convex combination

$$\frac{x_i+pc}{p+1} = \frac{1}{p+1}x_i + \frac{p}{p+1}c,$$

we obtain

$$\sum_{i=1}^{\infty}\lambda_i f\left(\frac{x_i+pc}{p+1}\right) \le \sum_{i=1}^{\infty}\lambda_i \frac{f(x_i)+pf(c)}{p+1}. \tag{4.9}$$

Since

$$f(c) = f\left(\sum_{i=1}^{\infty}\lambda_i x_i\right) \le \sum_{i=1}^{\infty}\lambda_i f(x_i),$$

it follows that

$$\sum_{i=1}^{\infty}\lambda_i \frac{f(x_i)+pf(c)}{p+1} = \frac{1}{p+1}\sum_{i=1}^{\infty}\lambda_i f(x_i) + \frac{p}{p+1}f(c) \le \sum_{i=1}^{\infty}\lambda_i f(x_i). \tag{4.10}$$

Let $\alpha a + \beta b$ be the convex combination such that $c = \alpha a + \beta b$. Since

$$c = \frac{b-c}{b-a}a + \frac{c-a}{b-a}b,$$

the coefficients are $\alpha = (b-c)/(b-a)$ and $\beta = (c-a)/(b-a)$. According to the right-hand side (containing the second and third members) of the inequality in formula (3.1), we have

$$\sum_{i=1}^{\infty}\lambda_i f(x_i) \le \alpha f(a) + \beta f(b) = \frac{b-c}{b-a}f(a) + \frac{c-a}{b-a}f(b). \tag{4.11}$$

The chain of the inequalities in formulae (4.8)-(4.11) yields formula (4.7). $\square$

Combining the inequalities in formula (4.7) and formula (4.6), we reach the full refinement of the double inequality in formula (3.1) as follows.

Corollary 4.2. *Let $\sum_{i=1}^{\infty}\lambda_i x_i$ be an infinite convex combination of points $x_i \in [a,b]$, let $c = \sum_{i=1}^{\infty}\lambda_i x_i$ be its center, and let $p \ge 0$ be a number.*

Then each convex function $f:[a,b]\to\mathbb{R}$ satisfies the multiple inequality

$$\begin{aligned} f\Big(\sum_{i=1}^{\infty}\lambda_i x_i\Big) &\le \sum_{i=1}^{\infty}\lambda_i f\Big(\frac{x_i+pc}{p+1}\Big) \\ &\le \sum_{i=1}^{\infty}\lambda_i \frac{f(x_i)+pf(c)}{p+1} \\ &\le \sum_{i=1}^{\infty}\lambda_i f(x_i) \\ &\le \sum_{i=1}^{\infty}\lambda_i \Big[\frac{b-x_i}{b-a} f\Big(\frac{pa+x_i}{p+1}\Big)+\frac{x_i-a}{b-a} f\Big(\frac{x_i+pb}{p+1}\Big)\Big] \\ &\le \sum_{i=1}^{\infty}\lambda_i \frac{p(b-x_i)f(a)+(b-a)f(x_i)+p(x_i-a)f(b)}{(p+1)(b-a)} \\ &\le \frac{b-c}{b-a}f(a)+\frac{c-a}{b-a}f(b). \end{aligned} \tag{4.12}$$

5. Infinite convex combination of infinite convex combinations

We observe the influence of a convex function to the infinite convex combination of infinite convex combinations.

Lemma 5.1. *Let $(\sum_{i=1}^{\infty}\lambda_{ij}x_{ij})_{j=1}^{\infty}$ be a sequence of infinite convex combinations of points $x_{ij}\in[a,b]$, and let $(\kappa_j)_{j=1}^{\infty}$ be a sequence of coefficients $\kappa_j\in[0,1]$ such that $\sum_{j=1}^{\infty}\kappa_j=1$.*

Then each convex function $f:[a,b]\to\mathbb{R}$ satisfies the double inequality

$$\begin{aligned} f\Big(\sum_{j=1}^{\infty}\kappa_j\sum_{i=1}^{\infty}\lambda_{ij}x_{ij}\Big) &\le \sum_{j=1}^{\infty}\kappa_j f\Big(\sum_{i=1}^{\infty}\lambda_{ij}x_{ij}\Big) \\ &\le \sum_{j=1}^{\infty}\kappa_j\sum_{i=1}^{\infty}\lambda_{ij}f(x_{ij}). \end{aligned} \tag{5.1}$$

Proof. Infinite convex combinations will be denoted as the points

$$x_j=\sum_{i=1}^{\infty}\lambda_{ij}x_{ij}$$

for $j=1,\dots,\infty$. Relying on Theorem A, we can conclude that points x_j and their infinite convex combination

$$x=\sum_{j=1}^{\infty}\kappa_j x_j=\sum_{j=1}^{\infty}\kappa_j\sum_{i=1}^{\infty}\lambda_{ij}x_{ij}$$

belong to $[a,b]$. Applying Jensen's inequality for infinite convex combinations in formula (3.2) to the above equalities, we get the double inequality

$$f(x)\le\sum_{j=1}^{\infty}\kappa_j f(x_j)\le\sum_{j=1}^{\infty}\kappa_j\sum_{i=1}^{\infty}\lambda_{ij}f(x_{ij}),$$

representing formula (5.1). ☐

By combining the sequences of infinite and binomial convex combinations, we can obtain the following.

Corollary 5.1. *Let* $(\sum_{i=1}^{\infty}\lambda_{ij}x_{ij})_{j=1}^{\infty}$ *be a sequence of infinite convex combinations of points* $x_{ij}\in[a,b]$, *let* $(\alpha_j a+\beta_j b)_{j=1}^{\infty}$ *be the sequence of convex combinations such that* $\alpha_j a+\beta_j b=\sum_{i=1}^{\infty}\lambda_{ij}x_{ij}$, *let* $(\kappa_j)_{j=1}^{\infty}$ *be a sequence of coefficients* $\kappa_j\in[0,1]$ *such that* $\sum_{j=1}^{\infty}\kappa_j=1$, *and let* $\alpha a+\beta b$ *be the convex combination such that* $\alpha a+\beta b=\sum_{j=1}^{\infty}\kappa_j(\alpha_j a+\beta_j b)$.

Then each convex function $f:[a,b]\to\mathbb{R}$ *satisfies the multiple inequality*

$$\begin{aligned} f(\alpha a+\beta b) &\leq \sum_{j=1}^{\infty}\kappa_j f(\alpha_j a+\beta_j b) \\ &\leq \sum_{j=1}^{\infty}\kappa_j\sum_{i=1}^{\infty}\lambda_{ij}f(x_{ij}) \\ &\leq \sum_{j=1}^{\infty}\kappa_j(\alpha_j f(a)+\beta_j f(b)) \\ &\leq \alpha f(a)+\beta f(b). \end{aligned} \tag{5.2}$$

Proof. The inequality in formula (5.2) can be derived using the inequalities in formula (3.2) and formula (3.1). □

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