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SOME PROPERTIES OF GENERALIZED NIELSEN'S β-FUNCTION WITH DOUBLE PARAMETERS

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Abstract. In this paper, we show complete monotonicity and convexity for (p, k) - generalized Nielsen's β-function with double parameters, and some inequalities are obtained. Moreover, the monotonic properties can be generalized to the m-order derivative of it.

1. Preliminaries and Introduction

A function f is said to be completely monotonic on an interval I if $f : I \to \mathbf{R}$ has derivatives of all orders on I and satisfies $(-1)^n f^{(n)}(x) \geq 0$ for $x \in I$ and $n \geq 0$. If $f : I \to \mathbf{R}$ is positive and $(-1)^n [\ln f(x)]^{(n)} \geq 0$, then we call f a logarithmically completely monotonic function. There are the following relations between the above functions:

(1) a function f is completely monotonic on $(0, \infty)$ if and only if it is a Laplace transform, that is, there is a positive measure v on $(0, \infty)$ such that

$$f(x) = \int_0^\infty e^{-xt} dv(t), x > 0.$$

(2) the set of all logarithmically completely monotonic functions is a strict subset of all completely monotonic functions. For the background and application, the readers may see ([1–5]).

The classical Nielsen's β-function can be defined as ([6–9, 13, 14])

$$\beta(x) = \int_0^1 \frac{t^{x-1}}{1+t} dt, \tag{1.1}$$

$$= \int_0^\infty \frac{e^{-xt}}{1+e^{-t}} dt, \tag{1.2}$$

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$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n+x}, \tag{1.3}$$

$$= \frac{1}{2}\left\{\psi\left(\frac{x+1}{2}\right) - \psi\left(\frac{x}{2}\right)\right\}, \tag{1.4}$$

where $x \in (0, \infty)$, $\psi(x) = \frac{d}{dx}\ln\Gamma(x)$ is the digamma or psi function and $\Gamma(x)$ is the Euler's Gamma function. It satisfies the following recursive relations:

$$\beta(x+1) = \frac{1}{x} - \beta(x), \tag{1.5}$$

$$\beta(x) + \beta(1-x) = \frac{\pi}{\sin \pi x}. \tag{1.6}$$

The Nielsen's β-function has been vastly researched (see [8–13]). Recently, K. Nantomah studied the inequalities and properties of a generalization of the Nielsen's function in [14]. In the paper, we will follow the techniques and procedures in [8] to study a (p, k)-generalization of the Nielsen's β-function function. The notations $\mathbf{N} = \{1,\ 2,\ 3,\ 4,\ \cdots\}$ and $\mathbf{N}_0 = \mathbf{N} \cup \{0\}$.

The Gamma function $\Gamma_{p,k}(x)$ can be defined as

$$\Gamma_{p,k}(x) = \int_0^p t^{x-1}\left(1 - \frac{t^k}{pk}\right)^p dt, \tag{1.7}$$

here $k,\ x \in (0, \infty)$, and $p \in \mathbf{N}$. Some properties of $\Gamma_{p,k}(x)$ has been proved in [15], such as $\Gamma_{p,k}(k) = 1$, $\Gamma_{p,k}(x+k) = \frac{pkx}{x+pk+k}\Gamma_{p,k}(x)$, and $\Gamma_{p,k}(ak) = \frac{p+1}{p}k^{a-1}\Gamma_{p,k}(x),\ a \in (0, \infty)$.

2. The generalization of the Nielsen's β-function

We discuss some properties and inequalities of a (p, k)-generalized Nielsen's β-function in this part.

Definition 2.1. The (p, k)-generalized Nielsen's β-function can be defined as

$$\beta_{p,k}(x) = \int_0^1 \frac{1 - t^{2k(p+1)}}{1 + t^k} t^{x-1} dt, \tag{2.1}$$

$$= \int_0^\infty \frac{1 - e^{-2k(p+1)t}}{1 + e^{-kt}} e^{-xt} dt, \tag{2.2}$$

$$= \sum_{n=0}^{p}\left(\frac{1}{2nk+x} - \frac{1}{2nk+k+x}\right), \tag{2.3}$$

$$= \frac{1}{2}\left\{\psi_{p,k}\left(\frac{x+k}{2}\right) - \psi_{p,k}\left(\frac{x}{2}\right)\right\}, \tag{2.4}$$

where $k,\ x \in (0, \infty)$, and $p \in \mathbf{N}$, $\psi_{p,k}(x) = \frac{d}{dx}\ln\Gamma_{p,k}(x)$ and $\lim\limits_{p\to\infty,\ k=1} \beta_{p,k}(x) = \beta(x)$.

Remark 2.1. Using series and integral representations of the function $\Gamma_{p,k}(x)$

$$\psi_{p,k}(x) = \frac{d}{dx}\ln\Gamma_{p,k} = \frac{1}{k}\ln(pk) - \sum_{n=0}^{p}\frac{1}{nk+x}$$

$$= \frac{1}{k}\ln(pk) - \int_0^\infty \frac{1 - e^{-k(p+1)t}}{1 - e^{-kt}} e^{-xt} dt,$$

we easily obtain

$$\begin{aligned}
\beta_{p,k}(x) &:= \frac{1}{2}\left\{\psi_{p,k}\left(\frac{x+k}{2}\right) - \psi_{p,k}\left(\frac{x}{2}\right)\right\} \\
&= \frac{1}{2}\left\{\frac{1}{k}\ln(pk) - \sum_{n=0}^{p}\frac{1}{nk+\frac{k+x}{2}} - \left(\frac{1}{k}\ln(pk) - \sum_{n=0}^{p}\frac{1}{nk+\frac{x}{2}}\right)\right\} \\
&= \sum_{n=0}^{p}\left(\frac{1}{2nk+x} - \frac{1}{2nk+k+x}\right) \\
&= \frac{1}{2}\left(\frac{1}{k}\ln(pk) - \int_0^\infty \frac{1-e^{-k(p+1)t}}{1-e^{-kt}} e^{-\frac{x+k}{2}t} dt\right) \\
&- \frac{1}{2}\left(\frac{1}{k}\ln(pk) - \int_0^\infty \frac{1-e^{-k(p+1)t}}{1-e^{-kt}} e^{-\frac{x}{2}t} dt\right) (u = \frac{t}{2}) \\
&= \int_0^\infty \frac{1-e^{-k(p+1)u}}{1-e^{-2ku}} e^{-ux} du - \int_0^\infty \frac{1-e^{-2k(p+1)u}}{1-e^{-2ku}} e^{-(x+k)u} du \\
&= \int_0^\infty \frac{1-e^{-2k(p+1)t}}{1+e^{-kt}} e^{-xt} dt (e^{-t} = u) \\
&= \int_0^1 \frac{1-t^{2k(p+1)}}{1+t^k} t^{x-1} dt.
\end{aligned}$$

Theorem 2.1. *For $x > 0$ and $m \in \boldsymbol{N}_0$, the (p,k)-generalized function $\beta_{p,k}(x)$ has the following properties:*
(1) $\beta_{p,k}^{(m)}(x)$ is decreasing and positive if m is even;
(2) $\beta_{p,k}^{(m)}(x)$ is increasing and negative if m is odd;
(3) $|\beta_{p,k}^{(m)}(x)|$ is decreasing for all m.

Proof. By (2.2), we can obtain

$$\beta_{p,k}^{(m)}(x) = \int_0^\infty (-1)^m t^m \frac{1-e^{-2k(p+1)t}}{1+e^{-kt}} e^{-xt} dt, \tag{2.5}$$

and

$$\left(\beta_{p,k}^{(m)}(x)\right)' = (-1)^{m+1} \int_0^\infty t^m \frac{1-e^{-2k(p+1)t}}{1+e^{-kt}} e^{-xt} dt. \tag{2.6}$$

If m is even, (2.5)≥ 0 and (2.6)≤ 0; if m is odd, (2.5)≤ 0 and (2.6)≥ 0.
So,

$$\left|\beta_{p,k}^{(m)}(x)\right|' = -\int_0^\infty t^m \frac{1-e^{-2k(p+1)t}}{1+e^{-kt}} e^{-xt} dt \leq 0. \tag{2.7}$$

Hence we achieve the conclusions. □

Theorem 2.2. *For $x > 0$ and $m \in \boldsymbol{N}_0$, the (p,k)-generalized function $\beta_{p,k}(x)$ has the following properties:*
(1) $\beta_{p,k}(x)$ is completely monotonic;

(2) $\beta_{p,k}^{(m)}(x)$ *is completely monotonic if* m *is even;*
(3) $-\beta_{p,k}^{(m)}(x)$ *is completely monotonic if* m *is odd.*

Proof. By (2.5), we get

$$(-1)^l \beta_{p,k}^{(m+l)}(x) = (-1)^{m+2l} \int_0^\infty t^{m+l} \frac{1-e^{-2k(p+1)t}}{1+e^{-kt}} e^{-xt} dt. \tag{2.8}$$

If m is even, (2.8)≥ 0; if m is odd, (2.8)≤ 0. By the definition of completely monotonic, we achieve the conclusions. □

Theorem 2.3. *For* $x > 0$ *and* $m \in \boldsymbol{N}_0$ *, the* (p, k)*-generalized function* $\beta_{p,k}(x)$ *has the following properties:*
(1) $\beta_{p,k}^{(m)}(x)$ *is convex if* m *is even;*
(2) $\beta_{p,k}^{(m)}(x)$ *is concave if* m *is odd;*
(3) $|\beta_{p,k}^{(m)}(x)|$ *is convex for all* m*.*

Proof. By (2.2), we get

$$\left(\beta_{p,k}^{(m)}(x)\right)'' = \beta_{p,k}^{(m+2)}(x) = (-1)^{m+2} \int_0^\infty t^{m+2} \frac{1-e^{-2k(p+1)t}}{1+e^{-kt}} e^{-xt} dt. \tag{2.9}$$

If m is even, (2.9)≥ 0; if m is odd, (2.9)≤ 0.

$$\left|\beta_{p,k}^{(m)}(x)\right|'' = (-1)^2 \int_0^\infty t^{m+2} \frac{1-e^{-2k(p+1)t}}{1+e^{-kt}} e^{-xt} dt \geq 0. \tag{2.10}$$

The proof is complete. □

Theorem 2.4. *The generalized function* $\beta_{p,k}(x)$ *satisfies the inequality*

$$\left|\beta_{p,k}^{(\frac{m}{a}+\frac{n}{b})}\left(\frac{x}{a}+\frac{y}{b}\right)\right| \leq \left|\beta_{p,k}^{(m)}(x)\right|^{\frac{1}{a}} \left|\beta_{p,k}^{(n)}(y)\right|^{\frac{1}{b}}, \tag{2.11}$$

for $a > 1,\ b > 1,\ x > 0,\ y > 0,\ m,\ n \in \boldsymbol{N}_0$ *and* $\frac{1}{a} + \frac{1}{b} = 1$*.*

Proof. By (2.5) and Hölder's inequality, we get

$$\begin{aligned}
&\left|\beta_{p,k}^{(\frac{m}{a}+\frac{n}{b})}\left(\frac{x}{a}+\frac{y}{b}\right)\right| \\
&= \left|\int_0^\infty (-1)^{\frac{m}{a}+\frac{n}{b}} t^{\frac{m}{a}+\frac{n}{b}} \frac{1-e^{-2k(p+1)t}}{1+e^{-kt}} e^{-(\frac{x}{a}+\frac{y}{b})t} dt\right| \\
&= \left|\int_0^\infty \left(\frac{1-e^{-2k(p+1)t}}{1+e^{-kt}} t^m e^{-xt}\right)^{\frac{1}{a}} \left(\frac{1-e^{-2k(p+1)t}}{1+e^{-kt}} t^n e^{-yt}\right)^{\frac{1}{b}} dt\right| \\
&\leq \left|\int_0^\infty \frac{1-e^{-2k(p+1)t}}{1+e^{-kt}} t^m e^{-xt} dt\right|^{\frac{1}{a}} \left|\int_0^\infty \frac{1-e^{-2k(p+1)t}}{1+e^{-kt}} t^n e^{-yt} dt\right|^{\frac{1}{b}} \\
&= \left|\beta_{p,k}^{(m)}(x)\right|^{\frac{1}{a}} \left|\beta_{p,k}^{(n)}(y)\right|^{\frac{1}{b}}.
\end{aligned}$$

□

Corollary 2.1. *When $m=n$ is even in Theorem 2.4, then $\beta_{p,k}(x)$ satisfies*

$$\beta_{p,k}^{(m)}\left(\frac{x}{a}+\frac{y}{b}\right)\le\left(\beta_{p,k}^{(m)}(x)\right)^{\frac{1}{a}}\left(\beta_{p,k}^{(m)}(y)\right)^{\frac{1}{b}},\tag{2.12}$$

which indicates $\beta_{p,k}(x)$ is logarithmically convex. The inequality (2.12) also can get from Theorem 2.3.

Particularly, if $a=b=2,\ x=y$ and $m=n+2$ in Theorem 2.4, we achieve the inequality

$$\left|\beta_{p,k}^{(m+1)}(x)\right|^2\le\left|\beta_{p,k}^{(m+2)}(x)\right|\left|\beta_{p,k}^{(m)}(x)\right|.\tag{2.13}$$

Hence, let $m=0$ in (2.13), we can get

$$\left(\beta'_{p,k}(x)\right)^2\le\beta''_{p,k}(x)\beta_{p,k}(x),\tag{2.14}$$

which indicates $\frac{\beta'_{p,k}(x)}{\beta_{p,k}(x)}$ is increasing.

Theorem 2.5. *The generalized function $\beta_{p,k}(x)$ satisfies*

$$\left(\left|\beta_{p,k}^{(m)}(x)\right|+\left|\beta_{p,k}^{(n)}(y)\right|\right)^{\frac{1}{b}}\le\left|\beta_{p,k}^{(m)}(x)\right|^{\frac{1}{b}}+\left|\beta_{p,k}^{(n)}(y)\right|^{\frac{1}{b}},\tag{2.15}$$

for $b>1,\ x>0,\ y>0,$ and $m,\ n\in\boldsymbol{N}_0$.

Proof. By (2.5) and the Minkowski's inequality, we get

$$\begin{aligned}
&\left(\left|\beta_{p,k}^{(m)}(x)\right|+\left|\beta_{p,k}^{(n)}(y)\right|\right)^{\frac{1}{b}}\\
&=\left(\int_0^\infty t^m\frac{1-e^{-2k(p+1)t}}{1+e^{-kt}}e^{-xt}dt+\int_0^\infty t^n\frac{1-e^{-2k(p+1)t}}{1+e^{-kt}}e^{-yt}dt\right)^{\frac{1}{b}}\\
&=\left\{\int_0^\infty\left[\left(\frac{1-e^{-2k(p+1)t}}{1+e^{-kt}}t^me^{-xt}\right)^{\frac{1}{b}}\right]^b+\left[\left(\frac{1-e^{-2k(p+1)t}}{1+e^{-kt}}t^ne^{-yt}dt\right)^{\frac{1}{b}}\right]^b dt\right\}^{\frac{1}{b}}\\
&\le\left\{\int_0^\infty\left[\left(\frac{1-e^{-2k(p+1)t}}{1+e^{-kt}}t^me^{-xt}\right)^{\frac{1}{b}}+\left(\frac{1-e^{-2k(p+1)t}}{1+e^{-kt}}t^ne^{-yt}dt\right)^{\frac{1}{b}}\right]^b dt\right\}^{\frac{1}{b}}\\
&\le\left(\int_0^\infty\frac{1-e^{-2k(p+1)t}}{1+e^{-kt}}t^me^{-xt}dt\right)^{\frac{1}{b}}+\left(\int_0^\infty\frac{1-e^{-2k(p+1)t}}{1+e^{-kt}}t^ne^{-yt}dt\right)^{\frac{1}{b}}\\
&=\left|\beta_{p,k}^{(m)}(x)\right|^{\frac{1}{b}}+\left|\beta_{p,k}^{(n)}(y)\right|^{\frac{1}{b}}.
\end{aligned}$$

□

Theorem 2.6. *The generalized function $\beta_{p,k}(x)$ satisfies*

$$\left[\beta_{p,k}^{(m)}(xy)\right]^2\le\beta_{p,k}^{(m)}(x)\beta_{p,k}^{(m)}(y)\tag{2.16}$$

for $x>1,\ y>1,$ and $m\in\boldsymbol{N}_0$.

Proof. If m is even, $\beta_{p,k}^{(m)}(x)$ is decreasing and positive by Theorem 2.1, hence we can get

$$0 < \beta_{p,k}^{(m)}(xy) \leq \beta_{p,k}^{(m)}(x),$$

and

$$0 < \beta_{p,k}^{(m)}(xy) \leq \beta_{p,k}^{(m)}(y),$$

for $x > 1$ and $y > 1$, so

$$\left[\beta_{p,k}^{(m)}(xy)\right]^2 \leq \beta_{p,k}^{(m)}(x)\beta_{p,k}^{(m)}(y).$$

If m is odd, $\beta_{p,k}^{(m)}(x)$ is increasing and negative, then

$$\beta_{p,k}^{(m)}(x) \leq \beta_{p,k}^{(m)}(xy) < 0,$$

and

$$\beta_{p,k}^{(m)}(y) \leq \beta_{p,k}^{(m)}(xy) < 0,$$

for $x > 1$ and $y > 1$, so

$$\left[\beta_{p,k}^{(m)}(xy)\right]^2 \leq \beta_{p,k}^{(m)}(x)\beta_{p,k}^{(m)}(y).$$

Hence we achieve the conclusions. □

The Theorem 2.6 can be generalized as:

Theorem 2.7. *The generalized function $\beta_{p,k}(x)$ satisfies*

$$\left|\beta_{p,k}^{(m)}\left(\prod_{i=1}^{n} x_i\right)\right|^n \leq \prod_{i=1}^{n}\left|\beta_{p,k}^{(m)}(x_i)\right| \tag{2.17}$$

for $m \in \boldsymbol{N}_0$, $n \in \boldsymbol{N}$ and $x_i > 1,\ i = 1,\ 2,\ \cdots,\ n$.

Proof. For $\prod_{i=1}^{n} x_i > x_l$ when $x_l > 1$, and we can get $|\beta_{p,k}^{(m)}(x)|$ is decreasing for all $m \in \mathbf{N}_0$ by Theorem 2.1, hence

$$0 < \left|\beta_{p,k}^{(m)}\left(\prod_{i=1}^{n} x_i\right)\right| \leq \left|\beta_{p,k}^{(m)}(x_l)\right|,\ l = 1,\ 2,\ \cdots,\ n.$$

Product the above inequalities, we can get (2.17). □

3. Conclusion

In this paper, we define a new two-parameter generalized Nielsen's β-function, obtain the function $\beta_{p,k}(x)$ is convex and completely monotonic. The completely monotonicity, convexity and some inequalities can be generalize to the m-order derivatives $\beta_{p,k}^{(m)}(x)$. In addition, the results are very important in evaluating some integrals.

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References

[1] R. L. Schilling, R. Song, and Z. Vondraček, *Bernstein Functions-Theory and Applications*, 2nd ed., de Gruyter Studies in Mathematics 37, Walter De Gruyter, Berlin, Germany, 2012.

[2] R. D. Atanassov and U. V. Tsoukrovski, *Some properties of a class of logarithmically completely monotonic functions*, C. R. Acad. Bulgare Sci., **41**(2) (1998), 21-23.

[3] C. Berg, *Intgral representation of some functions related to the gamma function*, Mediter. J. Math., **1**(4) (2004), 433-439.

[4] C. Berg, *Stieltjes-Pick-Bernstein-Schoenberg and their connection to complete monotonicity.* Pages 15-45. In: Positive definite functions. From Schoenberg to Space-Time Challenges. Ed. J. Mateu and E. Porcu. Dept. of Mathematics, University Jaume I, Castellon, Spain, 2008.

[5] E. Neuman, *Inequalities involving a logarithmially convex function and their applications to special functions*, J. Iequal. Pure Appl. Math., **7**(1) (2006) Art. 16.

[6] D. F. Connon, *On an integral involving the digamma function*, arXiv: 1212. 1432 [math.GM].

[7] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series and Products,* Edited by D. Zwillinger and V. Moll. Academic Press, New York, 8th Edition, 2014.

[8] K. Nantomah, *On Some Properties and Inequalities for the Nielsen's β Function*, arXiv: 1708. 06604vl 1432 [math.CA], 12 Pages.

[9] N. Nielsen, *Handbuch der Theorie der Gammafunktion*, First Edition, Leipzig: B. G. Teubner, 1906.

[10] K. Nantomah, *Monotoicity and Convexity Properties of the Nielsen's β-Function*, Probl. Anal. Issues Anal., **6**(24)(2) (2017), 81-93.

[11] K. Nantomah, *Monotoicity and convexity properties and some inequalities involving a generalized form of the Wallis' cosine formula*, Asian Research Journal of Mathematics, **6**(3) (2017), 1-10.

[12] K. N. Boyadzhiev, L. A.Medina, and V. H. Moll, *The integrals in Gradshteyn and Ryzhik, Part II: The incomplete beta function*, Scientia, Ser. A, Math. Sci., **18** (2009), 61-75.

[13] K. Nantomah, M. M. Iddrisu and C. A. Okpoti. *On a q-analogue of the Nielsen's β-Function,* Int. J. Math. And Appl., **6**(2-A)(2018), 163-171.

[14] K. Nantomah, *A generalization of the Nielsen's β-function*, Int. J. Open Problems Comput. Math., **11**(2) (2018), 16-26.

[15] K. Nantomah, E. Prempeh and S. B. Twum. *On a (p,k)-analogue of the Gamma function and some associated inequalities*, Moroccan J. Pure and Appl. Anal.(MJPAA), **2**(2) (2016), 79-90.

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