Turkish J. Ineq., 2 (2) (2018), Pages 11 –22 .

Turkish Journal of
INEQUALITIES

Available online at www.tjinequality.com

SYMMETRY OF CLASSICAL AND EXTENDED DYNAMIC INEQUALITIES UNIFIED ON TIME SCALE CALCULUS

MUHAMMAD JIBRIL SHAHAB SAHIR[1]

Abstract. The aim of this paper is to present some classical and dynamic inequalities such as Radon's Inequality, Bergström's Inequality, Rogers-Hölder's Inequality, Schlömilch's Inequality and Cauchy-Schwarz's Inequality in extended and symmetrical form to harmonize them on time scales.

1. Introduction

The inequality from (1.1) is called Bergström's Inequality in literature as given in [5–7,16].

Theorem 1.1. *If* $n \in \mathbb{N}$, $x_k \in \mathbb{R}$ *and* $y_k > 0$, $k \in \{1, 2, ..., n\}$, *then*

$$\frac{\left(\sum\limits_{k=1}^{n} x_k\right)^2}{\sum\limits_{k=1}^{n} y_k} \leq \sum_{k=1}^{n} \frac{x_k^2}{y_k}, \tag{1.1}$$

with equality if and only if $\frac{x_1}{y_1} = \frac{x_2}{y_2} = ... = \frac{x_n}{y_n}$.

The upcoming result is called Radon's Inequality as given in [19].

Theorem 1.2. *If* $n \in \mathbb{N}$, $x_k \geq 0$ *and* $y_k > 0$, $k \in \{1, 2, ..., n\}$ *and* $\beta \geq 0$, *then*

$$\frac{\left(\sum\limits_{k=1}^{n} x_k\right)^{\beta+1}}{\left(\sum\limits_{k=1}^{n} y_k\right)^{\beta}} \leq \sum_{k=1}^{n} \frac{x_k^{\beta+1}}{y_k^{\beta}}. \tag{1.2}$$

Inequality (1.2) is widely studied by many authors as it has many applications.

The following inequality is Radon's Inequality in generalized form as given in [11].

Key words and phrases. Bergström's Inequality, Radon's Inequality, Rogers-Hölder's Inequality, Schlömilch's Inequality, Time scales.

2010 *Mathematics Subject Classification.* Primary: 26D15. Secondary: 26D20, 34N05.

Received: 07/05/2018

Accepted: 19/12/2018.

Theorem 1.3. *If $n \in \mathbb{N}$, $x_k \geq 0$, $y_k > 0$, $k \in \{1, 2, ..., n\}$, $\beta \geq 0$ and $\gamma \geq 1$, then*

$$\frac{\left(\sum\limits_{k=1}^{n} x_k y_k^{\gamma-1}\right)^{\beta+\gamma}}{\left(\sum\limits_{k=1}^{n} y_k^{\gamma}\right)^{\beta+\gamma-1}} \leq \sum_{k=1}^{n} \frac{x_k^{\beta+\gamma}}{y_k^{\beta}}, \tag{1.3}$$

with equality if and only if $\frac{x_1}{y_1} = \frac{x_2}{y_2} = ... = \frac{x_n}{y_n}$.

We prove these results on time scales. The calculus of time scales was initiated by Stefan Hilger in [14]. A *time scale* is an arbitrary nonempty closed subset of the real numbers. The theory of time scales is applied to reveal the symmetry of continuous and discrete and to combine them in one comprehensive form. In time scale calculus, results are unified and extended. The time scale calculus is studied as delta calculus, nabla calculus and diamond-α calculus. This hybrid theory is also widely applied on dynamic inequalities. Basic work on dynamic inequalities is done by Ravi Agarwal, George Anastassiou, Martin Bohner, Allan Peterson, Donal O'Regan, Samir Saker and many other authors.

In this paper, it is assumed that all considerable integrals exist and are finite and $\mathbb{T}$ is a time scale, $a, b \in \mathbb{T}$ with $a < b$ and an interval $[a, b]_{\mathbb{T}}$ means the intersection of a real interval with the given time scale.

2. Preliminaries

We need here basic concepts of delta calculus. The results of delta calculus are adapted from [8, 9].

For $t \in \mathbb{T}$, the *forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$ is defined by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}.$$

The mapping $\mu : \mathbb{T} \to \mathbb{R}_0^+ = [0, +\infty)$ such that $\mu(t) := \sigma(t) - t$ is called the *forward graininess function*. The *backward jump operator* $\rho : \mathbb{T} \to \mathbb{T}$ is defined by

$$\rho(t) := \sup\{s \in \mathbb{T} : s < t\}.$$

The mapping $\nu : \mathbb{T} \to \mathbb{R}_0^+ = [0, +\infty)$ such that $\nu(t) := t - \rho(t)$ is called the *backward graininess function*. If $\sigma(t) > t$, we say that t is *right-scattered*, while if $\rho(t) < t$, we say that t is *left-scattered*. Also, if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, then t is called *right-dense*, and if $t > \inf \mathbb{T}$ and $\rho(t) = t$, then t is called *left-dense*. If $\mathbb{T}$ has a left-scattered maximum M, then $\mathbb{T}^k = \mathbb{T} - \{M\}$, otherwise $\mathbb{T}^k = \mathbb{T}$.

For a function $f : \mathbb{T} \to \mathbb{R}$, the delta derivative f^{Δ} is defined as follows:

Let $t \in \mathbb{T}^k$, if there exists $f^{\Delta}(t) \in \mathbb{R}$ such that for all $\epsilon > 0$, there exists a neighborhood U of t, such that

$$|f(\sigma(t)) - f(s) - f^{\Delta}(t)(\sigma(t) - s)| \leq \epsilon|\sigma(t) - s|,$$

for all $s \in U$, then f is said to be *delta differentiable* at t, and $f^{\Delta}(t)$ is called the *delta derivative* of f at t.

A function $f : \mathbb{T} \to \mathbb{R}$ is said to be *right-dense continuous* (*rd-continuous*), if it is continuous at each right-dense point and there exists a finite left limit at every left-dense point. The set of all rd-continuous functions is denoted by $C_{rd}(\mathbb{T}, \mathbb{R})$.

The next definition is given in [8, 9].

Definition 2.1. A function $F : \mathbb{T} \to \mathbb{R}$ is called a delta antiderivative of $f : \mathbb{T} \to \mathbb{R}$, provided that $F^{\Delta}(t) = f(t)$ holds for all $t \in \mathbb{T}^k$. Then the delta integral of f is defined by

$$\int_a^b f(t)\Delta t = F(b) - F(a).$$

Now we present a chain rule from [8].

Theorem 2.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be continuously differentiable and suppose $g : \mathbb{T} \to \mathbb{R}$ is delta differentiable. Then $f \circ g : \mathbb{T} \to \mathbb{R}$ is delta differentiable and the formula is*

$$(f \circ g)^{\Delta}(t) = g^{\Delta}(t) \left[\int_0^1 f'(g(t) + h\mu(t)g^{\Delta}(t))dh \right].$$

The following results of nabla calculus are taken from [4, 8, 9].

If $\mathbb{T}$ has a right-scattered minimum m, then $\mathbb{T}_k = \mathbb{T} - \{m\}$, otherwise $\mathbb{T}_k = \mathbb{T}$. A function $f : \mathbb{T}_k \to \mathbb{R}$ is called *nabla differentiable* at $t \in \mathbb{T}_k$, with nabla derivative $f^{\nabla}(t)$, if there exists $f^{\nabla}(t) \in \mathbb{R}$ such that for any $\epsilon > 0$, there exists a neighborhood V of t, such that

$$|f(\rho(t)) - f(s) - f^{\nabla}(t)(\rho(t) - s)| \leq \epsilon|\rho(t) - s|,$$

for all $s \in V$.

A function $f : \mathbb{T} \to \mathbb{R}$ is said to be *left-dense continuous* (*ld-continuous*), provided it is continuous at all left-dense points in $\mathbb{T}$ and its right-sided limits exist (finite) at all right-dense points in $\mathbb{T}$. The set of all ld-continuous functions is denoted by $C_{ld}(\mathbb{T}, \mathbb{R})$.

The next definition is given in [4, 8, 9].

Definition 2.2. A function $G : \mathbb{T} \to \mathbb{R}$ is called a nabla antiderivative of $g : \mathbb{T} \to \mathbb{R}$, provided that $G^{\nabla}(t) = g(t)$ holds for all $t \in \mathbb{T}_k$. Then the nabla integral of g is defined by

$$\int_a^b g(t)\nabla t = G(b) - G(a).$$

Lemma 2.1. [12]. *Let $f : \mathbb{R} \to \mathbb{R}$ be continuously differentiable and $g : \mathbb{T} \to \mathbb{R}$ be nabla differentiable, then $f \circ g$ is nabla differentiable and the formula is given by*

$$(f \circ g)^{\nabla}(t) = g^{\nabla}(t) \left[\int_0^1 f'(g(\rho(t)) + h\nu(t)g^{\nabla}(t))dh \right].$$

Now we present short introduction of the diamond-α derivative as given in [2, 23].

Let $\mathbb{T}$ be a time scale and $f(t)$ be differentiable on $\mathbb{T}$ in the Δ and ∇ senses. For $t \in \mathbb{T}_k^k$, where $\mathbb{T}_k^k = \mathbb{T}^k \cap \mathbb{T}_k$, the *diamond-$\alpha$ dynamic derivative* $f^{\diamond_\alpha}(t)$ is defined by

$$f^{\diamond_\alpha}(t) = \alpha f^{\Delta}(t) + (1 - \alpha)f^{\nabla}(t), \ 0 \leq \alpha \leq 1.$$

Thus f is diamond-α differentiable if and only if f is Δ and ∇ differentiable.

The diamond-α derivative reduces to the standard Δ-derivative for $\alpha = 1$, or the standard ∇-derivative for $\alpha = 0$. It represents a weighted dynamic derivative for $\alpha \in (0, 1)$.

Theorem 2.2. [23]. *Let $f, g : \mathbb{T} \to \mathbb{R}$ be diamond-α differentiable at $t \in \mathbb{T}$. Then*

(i) $f \pm g : \mathbb{T} \to \mathbb{R}$ *is diamond-*α *differentiable at* $t \in \mathbb{T}$*, with*

$$(f \pm g)^{\diamond_\alpha}(t) = f^{\diamond_\alpha}(t) \pm g^{\diamond_\alpha}(t).$$

(ii) $fg : \mathbb{T} \to \mathbb{R}$ *is diamond-*α *differentiable at* $t \in \mathbb{T}$*, with*

$$(fg)^{\diamond_\alpha}(t) = f^{\diamond_\alpha}(t)g(t) + \alpha f^{\sigma}(t)g^{\Delta}(t) + (1-\alpha)f^{\rho}(t)g^{\nabla}(t).$$

(iii) *For* $g(t)g^{\sigma}(t)g^{\rho}(t) \neq 0$, $\frac{f}{g} : \mathbb{T} \to \mathbb{R}$ *is diamond-*α *differentiable at* $t \in \mathbb{T}$*, with*

$$\left(\frac{f}{g}\right)^{\diamond_\alpha}(t) = \frac{f^{\diamond_\alpha}(t)g^{\sigma}(t)g^{\rho}(t) - \alpha f^{\sigma}(t)g^{\rho}(t)g^{\Delta}(t) - (1-\alpha)f^{\rho}(t)g^{\sigma}(t)g^{\nabla}(t)}{g(t)g^{\sigma}(t)g^{\rho}(t)}.$$

Definition 2.3. [23]. Let $a, t \in \mathbb{T}$ and $h : \mathbb{T} \to \mathbb{R}$. Then the diamond-$\alpha$ integral from a to t of h is defined by

$$\int_a^t h(s) \diamond_\alpha s = \alpha \int_a^t h(s)\Delta s + (1-\alpha)\int_a^t h(s)\nabla s, \ 0 \leq \alpha \leq 1,$$

provided that there exist delta and nabla integrals of h on $\mathbb{T}$.

Theorem 2.3. [23]. *Let* $a, b, t \in \mathbb{T}$*,* $c \in \mathbb{R}$*. Assume that* $f(s)$ *and* $g(s)$ *are* $\diamond_\alpha$*-integrable functions on* $[a, b]_{\mathbb{T}}$*. Then*

(i) $\int_a^t [f(s) \pm g(s)] \diamond_\alpha s = \int_a^t f(s) \diamond_\alpha s \pm \int_a^t g(s) \diamond_\alpha s$;
(ii) $\int_a^t cf(s) \diamond_\alpha s = c\int_a^t f(s) \diamond_\alpha s$;
(iii) $\int_a^t f(s) \diamond_\alpha s = -\int_t^a f(s) \diamond_\alpha s$;
(iv) $\int_a^t f(s) \diamond_\alpha s = \int_a^b f(s) \diamond_\alpha s + \int_b^t f(s) \diamond_\alpha s$;
(v) $\int_a^a f(s) \diamond_\alpha s = 0$.

We need the following results.

Definition 2.4. [10]. A function $f : \mathbb{T} \to \mathbb{R}$ is called convex on $I_{\mathbb{T}} = I \cap \mathbb{T}$, where I is an interval of $\mathbb{R}$ (open or closed), if

$$f(\lambda t + (1-\lambda)s) \leq \lambda f(t) + (1-\lambda)f(s), \tag{2.1}$$

for all $t, s \in I_{\mathbb{T}}$ and all $\lambda \in [0, 1]$ such that $\lambda t + (1-\lambda)s \in I_{\mathbb{T}}$.

The function f is strictly convex on $I_{\mathbb{T}}$ if (2.1) is strict for distinct $t, s \in I_{\mathbb{T}}$ and $\lambda \in (0, 1)$.

The function f is concave (respectively, strictly concave) on $I_{\mathbb{T}}$, if $-f$ is convex (respectively, strictly convex).

Theorem 2.4. [2]. *Let* $a, b \in \mathbb{T}$ *and* $c, d \in \mathbb{R}$*. Suppose that* $g \in C([a, b]_{\mathbb{T}}, (c, d))$ *and* $h \in C([a, b]_{\mathbb{T}}, \mathbb{R})$ *with* $\int\limits_a^b |h(s)| \diamond_\alpha s > 0$*. If* $\Phi \in C((c, d), \mathbb{R})$ *is convex, then generalized Jensen's Inequality is*

$$\Phi\left(\frac{\int\limits_a^b |h(s)|g(s) \diamond_\alpha s}{\int\limits_a^b |h(s)| \diamond_\alpha s}\right) \leq \frac{\int\limits_a^b |h(s)|\Phi(g(s)) \diamond_\alpha s}{\int\limits_a^b |h(s)| \diamond_\alpha s}. \tag{2.2}$$

If Φ *is strictly convex, then the inequality* $\leq$ *can be replaced by* $<$.

Example 2.1. [2]. If we set $\mathbb{T} = q^{\mathbb{N}_0}$ for $q > 1$ and $m < n$, then

$$\int_{q^m}^{q^n} f(x) \diamond_\alpha x = (q-1) \sum_{i=m}^{n-1} q^i [\alpha f(q^i) + (1-\alpha) f(q^{i+1})], \tag{2.3}$$

for $m, n \in \mathbb{N}_0$, where $\mathbb{N}_0$ is the set of nonnegative integers.

3. Main Results

In order to present our main results, first we present an extension of Radon's Inequality by applying Jensen's Inequality via time scales.

Theorem 3.1. *Let $w, f, g \in C([a,b]_{\mathbb{T}}, \mathbb{R})$ be $\diamond_\alpha$-integrable functions, where $w(x), g(x) \neq 0$.*

(*i*) *If $\beta \geq 0$ and $\gamma \geq 1$, then*

$$\frac{\left(\int_a^b |w(x)||f(x)||g(x)|^{\gamma-1} \diamond_\alpha x\right)^{\beta+\gamma}}{\left(\int_a^b |w(x)||g(x)|^{\gamma} \diamond_\alpha x\right)^{\beta+\gamma-1}} \leq \int_a^b \frac{|w(x)||f(x)|^{\beta+\gamma}}{|g(x)|^{\beta}} \diamond_\alpha x. \tag{3.1}$$

(*ii*) *If $0 < \beta + \gamma < 1$, then*

$$\frac{\left(\int_a^b |w(x)||f(x)||g(x)|^{\gamma-1} \diamond_\alpha x\right)^{\beta+\gamma}}{\left(\int_a^b |w(x)||g(x)|^{\gamma} \diamond_\alpha x\right)^{\beta+\gamma-1}} \geq \int_a^b \frac{|w(x)||f(x)|^{\beta+\gamma}}{|g(x)|^{\beta}} \diamond_\alpha x. \tag{3.2}$$

Equality holds in (3.1) and (3.2) if and only if $f(x) = cg(x)$, where c is a real constant.

Proof. We consider a function $\Phi : [0, \infty) \to \mathbb{R}$ defined by $\Phi(x) = x^{\beta+\gamma}$, $x \in [0, \infty)$, which is convex for $\beta \geq 0$ and $\gamma \geq 1$. Further we replace $|h(x)|$ by $|w(x)||g(x)|^\gamma$ and $|g(x)|$ by $\left|\frac{f(x)}{g(x)}\right|$, then Jensen's Inequality given in (2.2) takes the form

$$\left(\int_a^b \frac{|w(x)||g(x)|^\gamma}{\int_a^b |w(x)||g(x)|^\gamma \diamond_\alpha x} \left|\frac{f(x)}{g(x)}\right| \diamond_\alpha x\right)^{\beta+\gamma} \leq \frac{\int_a^b |w(x)||g(x)|^\gamma \left|\frac{f(x)}{g(x)}\right|^{\beta+\gamma} \diamond_\alpha x}{\int_a^b |w(x)||g(x)|^\gamma \diamond_\alpha x}.$$

Then,

$$\frac{\left(\int_a^b |w(x)||f(x)||g(x)|^{\gamma-1} \diamond_\alpha x\right)^{\beta+\gamma}}{\left(\int_a^b |w(x)||g(x)|^{\gamma} \diamond_\alpha x\right)^{\beta+\gamma}} \leq \frac{\int_a^b \frac{|w(x)||f(x)|^{\beta+\gamma}}{|g(x)|^{\beta}} \diamond_\alpha x}{\int_a^b |w(x)||g(x)|^\gamma \diamond_\alpha x}. \tag{3.3}$$

We multiply both sides of (3.3) by $\int_a^b |w(x)||g(x)|^\gamma \diamond_\alpha x$ and get the required result as given in (3.1).

The inequality given in (3.1) is reversed for $0 < \beta + \gamma < 1$.

It is clear that equality holds in (3.1) and (3.2) if and only if $f(x) = cg(x)$.

Now the proof of Theorem 3.1 is completed. □

Remark 3.1. If we set $\alpha = 1$, $\mathbb{T} = \mathbb{Z}$, $w(x) = 1$, $\beta = 1$, $\gamma = 1$, $f(k) = x_k \in \mathbb{R}$ and $g(k) = y_k \in (0, \infty)$ for $k \in \{1, 2, ..., n\}$, $n \in \mathbb{N}$, then (3.1) reduces to (1.1).

Remark 3.2. If we set $\alpha = 1$, $\mathbb{T} = \mathbb{Z}$, $w(x) = 1$, $\gamma = 1$, $f(k) = x_k \in [0, \infty)$ and $g(k) = y_k \in (0, \infty)$ for $k \in \{1, 2, ..., n\}$, $n \in \mathbb{N}$, then (3.1) reduces to (1.2) and we get reverse version of (1.2) for $-1 < \beta < 0$.

Remark 3.3. If we set $\alpha = 1$, $\mathbb{T} = \mathbb{Z}$, $w(x) = 1$, $f(k) = x_k \in [0, \infty)$ and $g(k) = y_k \in (0, \infty)$ for $k \in \{1, 2, ..., n\}$, $n \in \mathbb{N}$, then discrete version of (3.1) reduces to (1.3) and we get reverse discrete version of (1.3) for $0 < \beta + \gamma < 1$.

If we set $\mathbb{T} = \mathbb{R}$, then continuous version of (3.1) takes the form

$$\frac{\left(\int_a^b |w(x)||f(x)||g(x)|^{\gamma-1} dx\right)^{\beta+\gamma}}{\left(\int_a^b |w(x)||g(x)|^{\gamma} dx\right)^{\beta+\gamma-1}} \leq \int_a^b \frac{|w(x)||f(x)|^{\beta+\gamma}}{|g(x)|^{\beta}} dx.$$

Example 3.1. If we set $[a, b]_{\mathbb{T}} = [q^m, q^n]_{q^{\mathbb{N}_0}}$ for $q > 1$ and $m < n$, where $m, n \in \mathbb{N}_0$.

(*i*) If $\beta \geq 0$ and $\gamma \geq 1$, then (3.1) takes the form

$$\frac{\left[\sum\limits_{i=m}^{n-1} q^i \left\{\alpha|w(q^i)||f(q^i)||g(q^i)|^{\gamma-1} + (1-\alpha)|w(q^{i+1})||f(q^{i+1})||g(q^{i+1})|^{\gamma-1}\right\}\right]^{\beta+\gamma}}{\left[\sum\limits_{i=m}^{n-1} q^i \left\{\alpha|w(q^i)||g(q^i)|^{\gamma} + (1-\alpha)|w(q^{i+1})||g(q^{i+1})|^{\gamma}\right\}\right]^{\beta+\gamma-1}}$$
$$\leq \sum_{i=m}^{n-1} q^i \left\{\alpha \frac{|w(q^i)||f(q^i)|^{\beta+\gamma}}{|g(q^i)|^{\beta}} + (1-\alpha)\frac{|w(q^{i+1})||f(q^{i+1})|^{\beta+\gamma}}{|g(q^{i+1})|^{\beta}}\right\}.$$

(*ii*) If $0 < \beta + \gamma < 1$, then (3.2) takes the form

$$\frac{\left[\sum\limits_{i=m}^{n-1} q^i \left\{\alpha|w(q^i)||f(q^i)||g(q^i)|^{\gamma-1} + (1-\alpha)|w(q^{i+1})||f(q^{i+1})||g(q^{i+1})|^{\gamma-1}\right\}\right]^{\beta+\gamma}}{\left[\sum\limits_{i=m}^{n-1} q^i \left\{\alpha|w(q^i)||g(q^i)|^{\gamma} + (1-\alpha)|w(q^{i+1})||g(q^{i+1})|^{\gamma}\right\}\right]^{\beta+\gamma-1}}$$
$$\geq \sum_{i=m}^{n-1} q^i \left\{\alpha \frac{|w(q^i)||f(q^i)|^{\beta+\gamma}}{|g(q^i)|^{\beta}} + (1-\alpha)\frac{|w(q^{i+1})||f(q^{i+1})|^{\beta+\gamma}}{|g(q^{i+1})|^{\beta}}\right\}.$$

Upcoming result is symmetric Rogers-Hölder's Inequality and its reverse version. Its delta version is also given in [15].

Corollary 3.1. *Let $w, f, g \in C([a, b]_{\mathbb{T}}, \mathbb{R})$ be $\diamond_\alpha$-integrable functions.*

(*i*) *If $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$ with $p > 0, q > 0, r > 0$; $p < 0, q > 0, r < 0$ or $p > 0, q < 0, r < 0$, then*

$$\left(\int_a^b |w(x)||f(x)g(x)|^r \diamond_\alpha x\right)^{\frac{1}{r}}$$
$$\leq \left(\int_a^b |w(x)||f(x)|^p \diamond_\alpha x\right)^{\frac{1}{p}} \left(\int_a^b |w(x)||g(x)|^q \diamond_\alpha x\right)^{\frac{1}{q}}. \quad (3.4)$$

(ii) If $\frac{1}{p}+\frac{1}{q}=\frac{1}{r}$ *with* $p<0, q<0, r<0$*;* $p>0, q<0, r>0$ *or* $p<0, q>0, r>0$*, then*

$$\left(\int_a^b |w(x)||f(x)g(x)|^r \diamond_\alpha x\right)^{\frac{1}{r}} \geq \left(\int_a^b |w(x)||f(x)|^p \diamond_\alpha x\right)^{\frac{1}{p}} \left(\int_a^b |w(x)||g(x)|^q \diamond_\alpha x\right)^{\frac{1}{q}}. \quad (3.5)$$

Proof. Inequalities (3.4) and (3.5) are trivially true in the case when any of w, f or g is identically zero. We prove (3.4) and (3.5) for $w, f, g \in C([a,b]_{\mathbb{T}}, \mathbb{R}-\{0\})$. First we prove (i).

Case (a): If $p>0, q>0, r>0$.

Set $\beta>0$, $\gamma=1$ and $\beta+1=\frac{p}{r}>1$. Then result given in (3.1) takes the form

$$\frac{\left(\int_a^b |w(x)||f(x)| \diamond_\alpha x\right)^{\frac{p}{r}}}{\left(\int_a^b |w(x)||g(x)| \diamond_\alpha x\right)^{\frac{p}{r}-1}} \leq \int_a^b \frac{|w(x)||f(x)|^{\frac{p}{r}}}{|g(x)|^{\frac{p}{r}-1}} \diamond_\alpha x. \quad (3.6)$$

We replace $|g(x)|$ by $|g(x)|^q$ in (3.6), then

$$\frac{\left(\int_a^b |w(x)||f(x)| \diamond_\alpha x\right)^{\frac{p}{r}}}{\left(\int_a^b |w(x)||g(x)|^q \diamond_\alpha x\right)^{\frac{p}{r}-1}} \leq \int_a^b \frac{|w(x)||f(x)|^{\frac{p}{r}}}{|g(x)|^{q\left(\frac{p}{r}-1\right)}} \diamond_\alpha x. \quad (3.7)$$

Now taking power $\frac{1}{p}>0$ and replacing $|f(x)|$ by $|f(x)g(x)|^r$, then (3.7) takes the form

$$\left(\int_a^b |w(x)||f(x)g(x)|^r \diamond_\alpha x\right)^{\frac{1}{r}} \leq \left(\int_a^b |w(x)||f(x)|^p |g(x)|^{p-q\left(\frac{p}{r}-1\right)} \diamond_\alpha x\right)^{\frac{1}{p}} \left(\int_a^b |w(x)||g(x)|^q \diamond_\alpha x\right)^{\frac{1}{r}-\frac{1}{p}}. \quad (3.8)$$

Apply $\frac{r}{p}+\frac{r}{q}=1$ and (3.8) reduces to (3.4).

Case (b): If $p<0, q>0, r<0$.

Set $\beta>0$, $\gamma=1$ and $\beta+1=\frac{r}{p}>1$. Then result given in (3.1) becomes

$$\frac{\left(\int_a^b |w(x)||f(x)| \diamond_\alpha x\right)^{\frac{r}{p}}}{\left(\int_a^b |w(x)||g(x)| \diamond_\alpha x\right)^{\frac{r}{p}-1}} \leq \int_a^b \frac{|w(x)||f(x)|^{\frac{r}{p}}}{|g(x)|^{\frac{r}{p}-1}} \diamond_\alpha x. \quad (3.9)$$

We replace $|f(x)|$ by $|f(x)|^p$ in (3.9), then

$$\frac{\left(\int_a^b |w(x)||f(x)|^p \diamond_\alpha x\right)^{\frac{r}{p}}}{\left(\int_a^b |w(x)||g(x)| \diamond_\alpha x\right)^{\frac{r}{p}-1}} \leq \int_a^b \frac{|w(x)||f(x)|^r}{|g(x)|^{\frac{r}{p}-1}} \diamond_\alpha x. \quad (3.10)$$

Now taking power $\frac{1}{r} < 0$ and replacing $|g(x)|$ by $|g(x)|^q$, inequality (3.10) becomes

$$\left(\int_a^b |w(x)||f(x)|^p \diamond_\alpha x\right)^{\frac{1}{p}} \geq \left(\int_a^b |w(x)||f(x)|^r |g(x)|^{q\left(1-\frac{r}{p}\right)} \diamond_\alpha x\right)^{\frac{1}{r}} \left(\int_a^b |w(x)||g(x)|^q \diamond_\alpha x\right)^{\frac{1}{p}-\frac{1}{r}}. \quad (3.11)$$

Apply $\frac{r}{p} + \frac{r}{q} = 1$, so we get our required result.

Case (c): If $p > 0, q < 0, r < 0$.

Set $\beta > 0$, $\gamma = 1$ and $\beta + 1 = \frac{-p}{q} > 1$. Replacing $|w(x)|$ by $|w(x)||g(x)|^{-\frac{p}{q}-1}$, $|f(x)|$ by $|f(x)|^{-q}$, $|g(x)|$ by $|f(x)g(x)|^{\frac{-qr}{p}}$ and taking power $-\frac{1}{p} < 0$ in (3.1), we get our required result.

Now we prove reverse symmetric Rogers-Hölder's Inequality given in (ii) for all cases.

Case (a): If $p < 0, q < 0, r < 0$, then proof of (3.5) is similar to the proof of Case: (a) of (3.4).

Case (b): If $p > 0, q < 0, r > 0$, then proof of (3.5) is similar to the proof of Case: (b) of (3.4).

Case (c): If $p < 0, q > 0, r > 0$, then (3.5) follows by a similar proof as given in Case: (c) of (3.4).

Thus, the proof is complete. □

Remark 3.4. If $\alpha = 1$, $\mathbb{T} = \mathbb{Z}$, $w(k) = \lambda_k > 0$ for $k \in \{1, 2, ..., n\}$ with $\sum_{k=1}^{n} \lambda_k = 1$, $f(k) = a_k \in \mathbb{C} - \{0\}$ and $g(k) = b_k \in \mathbb{C} - \{0\}$ for $k \in \{1, 2, ..., n\}$, then we get discrete versions of (3.4) and (3.5) as given in [17, page 19].

Corollary 3.2. *Let $w, f_i \in C([a, b]_{\mathbb{T}}, \mathbb{R} - \{0\})$ for $i = 1, 2, 3$ be $\diamond_\alpha$-integrable functions. Further assume that $\left|\prod_{i=1}^{3} f_i\right| = M$ for some positive real number M. Let $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 0$ for three nonzero real numbers p, q, r.*

(i) If all but one of p, q, r are positive, then

$$\left(\int_a^b |w(x)||f_1(x)|^p \diamond_\alpha x\right)^{\frac{1}{p}} \left(\int_a^b |w(x)||f_2(x)|^q \diamond_\alpha x\right)^{\frac{1}{q}} \times \left(\int_a^b |w(x)||f_3(x)|^r \diamond_\alpha x\right)^{\frac{1}{r}} \geq M. \quad (3.12)$$

(ii) If all but one of p, q, r are negative, then

$$\left(\int_a^b |w(x)||f_1(x)|^p \diamond_\alpha x\right)^{\frac{1}{p}} \left(\int_a^b |w(x)||f_2(x)|^q \diamond_\alpha x\right)^{\frac{1}{q}} \times \left(\int_a^b |w(x)||f_3(x)|^r \diamond_\alpha x\right)^{\frac{1}{r}} \leq M. \quad (3.13)$$

Proof. To prove part (i), let $p > 0$, $q > 0$, $r < 0$ and we rearrange $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 0$ as $\frac{1}{(-\frac{p}{r})} + \frac{1}{(-\frac{q}{r})} = 1$, where $-\frac{p}{r} > 1$.

Set $\beta > 0$, $\gamma = 1$ and $\beta + \gamma = -\frac{p}{r}$ for $p > 0$, $q > 0$, $r < 0$. Then result given in (3.1) becomes

$$\frac{\left(\int_a^b |w(x)||f_1(x)| \diamond_\alpha x\right)^{-\frac{p}{r}}}{\left(\int_a^b |w(x)||f_2(x)| \diamond_\alpha x\right)^{-\frac{p}{r}-1}} \leq \int_a^b \frac{|w(x)||f_1(x)|^{-\frac{p}{r}}}{|f_2(x)|^{-\frac{p}{r}-1}} \diamond_\alpha x. \tag{3.14}$$

We replace $|f_2(x)|$ by $|f_2(x)|^q$ in (3.14) and get

$$\frac{\left(\int_a^b |w(x)||f_1(x)| \diamond_\alpha x\right)^{-\frac{p}{r}}}{\left(\int_a^b |w(x)||f_2(x)|^q \diamond_\alpha x\right)^{-\frac{p}{r}-1}} \leq \int_a^b \frac{|w(x)||f_1(x)|^{-\frac{p}{r}}}{|f_2(x)|^{q(-\frac{p}{r}-1)}} \diamond_\alpha x. \tag{3.15}$$

Now taking power $\frac{1}{p} > 0$ and replacing $|f_1(x)|$ by $|f_1(x)f_2(x)|^{-r}$, inequality (3.15) takes the form

$$\left(\int_a^b |w(x)||f_1(x)f_2(x)|^{-r} \diamond_\alpha x\right)^{-\frac{1}{r}}$$
$$\leq \left(\int_a^b |w(x)||f_1(x)|^p|f_2(x)|^{p+q(\frac{p}{r}+1)} \diamond_\alpha x\right)^{\frac{1}{p}} \left(\int_a^b |w(x)||f_2(x)|^q \diamond_\alpha x\right)^{-\frac{1}{r}-\frac{1}{p}}. \tag{3.16}$$

Applying $\left|\prod_{i=1}^{3} f_i\right| = M$ for some positive real number M and $\frac{1}{p}+\frac{1}{q}+\frac{1}{r} = 0$ for three nonzero real numbers p, q, r, inequality (3.16) reduces to (3.12).

Proof of part (ii) is similar to the proof of part (i). □

Remark 3.5. If $r = -1$ and $M = 1$, then (3.12) becomes

$$\int_a^b |w(x)||f_1(x)f_2(x)| \diamond_\alpha x$$
$$\leq \left(\int_a^b |w(x)||f_1(x)|^p \diamond_\alpha x\right)^{\frac{1}{p}} \left(\int_a^b |w(x)||f_2(x)|^q \diamond_\alpha x\right)^{\frac{1}{q}}, \tag{3.17}$$

which is Rogers-Hölder's Inequality as given in [2,18].

Remark 3.6. If $\alpha = 1$, $\mathbb{T} = \mathbb{Z}$, $w(x) = 1$ and $M = 1$, then we get discrete versions of (3.12) and (3.13) for three sets of positive values $f_1(k) = x_k$, $f_2(k) = y_k$ and $f_3(k) = z_k$ for $k \in \{1, 2, ..., n\}$ as given in [1, page 147].

The upcoming inequality is called Schlömilch's Inequality in literature. We present here direct proof of Schlömilch's Inequality from extended Radon's Inequality on time scales.

Corollary 3.3. *Let $w, f \in C([a, b]_{\mathbb{T}}, \mathbb{R})$ be $\diamond_\alpha$-integrable functions with $\int_a^b |w(x)| \diamond_\alpha x = 1$, where $w(x) \neq 0$. If $\eta_2 \geq \eta_1 > 0$, then*

$$\left(\int_a^b |w(x)||f(x)|^{\eta_1} \diamond_\alpha x\right)^{\frac{1}{\eta_1}} \leq \left(\int_a^b |w(x)||f(x)|^{\eta_2} \diamond_\alpha x\right)^{\frac{1}{\eta_2}}. \tag{3.18}$$

Proof. Set $\beta+\gamma=\frac{\eta_2}{\eta_1}\geq 1$ and $g(x)\equiv 1$ in (3.1), we get

$$\frac{\left(\int_a^b |w(x)||f(x)|\diamond_\alpha x\right)^{\frac{\eta_2}{\eta_1}}}{\left(\int_a^b |w(x)|\diamond_\alpha x\right)^{\frac{\eta_2}{\eta_1}-1}} \leq \int_a^b |w(x)||f(x)|^{\frac{\eta_2}{\eta_1}}\diamond_\alpha x. \tag{3.19}$$

As $\int_a^b |w(x)|\diamond_\alpha x=1$, inequality (3.19) becomes

$$\left(\int_a^b |w(x)||f(x)|\diamond_\alpha x\right)^{\frac{\eta_2}{\eta_1}} \leq \int_a^b |w(x)||f(x)|^{\frac{\eta_2}{\eta_1}}\diamond_\alpha x. \tag{3.20}$$

Replacing $|f(x)|$ by $|f(x)|^{\eta_1}$ and taking power $\frac{1}{\eta_2}$, inequality (3.20) gives our required claim. □

Corollary 3.4. *Let $w,f,g\in C([a,b]_{\mathbb{T}},\mathbb{R})$ be $\diamond_\alpha$-integrable functions with $\int_a^b |w(x)|\diamond_\alpha x=1$, where $w(x)\neq 0$. If $\frac{1}{p}+\frac{1}{q}<1$ with $p>1$ and $q>1$, then*

$$\int_a^b |w(x)||f(x)g(x)|\diamond_\alpha x \leq \left(\int_a^b |w(x)||f(x)|^p\diamond_\alpha x\right)^{\frac{1}{p}}\left(\int_a^b |w(x)||g(x)|^q\diamond_\alpha x\right)^{\frac{1}{q}}.$$

Proof. Let $\varphi:=\frac{1}{p}+\frac{1}{q}<1$, $\zeta_1=\varphi p<p$ and $\zeta_2=\varphi q<q$. Then $\sum_{i=1}^{2}\frac{1}{\zeta_i}=1$, where $\zeta_i>1$.

The rest of proof follows by applying Rogers-Hölder's Inequality as given in (3.17) and by Schlömilch's Inequality as given in (3.18). □

Now we present Theorem 3.1 in two dimensions by applying Jensen's Inequality via time scales. The diamond-α integral for a function of two variables is defined in [2,3].

Theorem 3.2. *Let $w,f,g\in C([a,b]_{\mathbb{T}}\times[a,b]_{\mathbb{T}},\mathbb{R})$ be $\diamond_\alpha$-integrable functions, where $w(x,y)$, $g(x,y)\neq 0$.*

(i) If $\beta\geq 0$ and $\gamma\geq 1$, then

$$\frac{\left(\int_a^b\int_a^b |w(x,y)||f(x,y)||g(x,y)|^{\gamma-1}\diamond_\alpha x\diamond_\alpha y\right)^{\beta+\gamma}}{\left(\int_a^b\int_a^b |w(x,y)||g(x,y)|^{\gamma}\diamond_\alpha x\diamond_\alpha y\right)^{\beta+\gamma-1}} \leq \int_a^b\int_a^b \frac{|w(x,y)||f(x,y)|^{\beta+\gamma}}{|g(x,y)|^{\beta}}\diamond_\alpha x\diamond_\alpha y. \tag{3.21}$$

(ii) If $0<\beta+\gamma<1$, then

$$\frac{\left(\int_a^b\int_a^b |w(x,y)||f(x,y)||g(x,y)|^{\gamma-1}\diamond_\alpha x\diamond_\alpha y\right)^{\beta+\gamma}}{\left(\int_a^b\int_a^b |w(x,y)||g(x,y)|^{\gamma}\diamond_\alpha x\diamond_\alpha y\right)^{\beta+\gamma-1}} \geq \int_a^b\int_a^b \frac{|w(x,y)||f(x,y)|^{\beta+\gamma}}{|g(x,y)|^{\beta}}\diamond_\alpha x\diamond_\alpha y. \tag{3.22}$$

Equality holds in (3.21) and (3.22) if and only if $f(x,y)=cg(x,y)$, where c is a real constant.

Proof. Similar to the proof of Theorem 3.1. □

Remark 3.7. Set $\beta = 1$, $\gamma = 1$ and $\beta + \gamma = 2$. Then (3.21) takes the form

$$\frac{\left(\int_a^b \int_a^b |w(x,y)||f(x,y)| \diamond_\alpha x \diamond_\alpha y\right)^2}{\int_a^b \int_a^b |w(x,y)||g(x,y)| \diamond_\alpha x \diamond_\alpha y} \leq \int_a^b \int_a^b \frac{|w(x,y)||f(x,y)|^2}{|g(x,y)|} \diamond_\alpha x \diamond_\alpha y. \tag{3.23}$$

Now we replace $|w(x,y)|$ by $|w(x,y)g(x,y)|$, inequality (3.23) takes the form

$$\int_a^b \int_a^b |w(x,y)||f(x,y)||g(x,y)| \diamond_\alpha x \diamond_\alpha y$$
$$\leq \sqrt{\left(\int_a^b \int_a^b |w(x,y)||f(x,y)|^2 \diamond_\alpha x \diamond_\alpha y\right)} \sqrt{\left(\int_a^b \int_a^b |w(x,y)||g(x,y)|^2 \diamond_\alpha x \diamond_\alpha y\right)}. \tag{3.24}$$

The inequality (3.24) is called Cauchy-Schwarz's Inequality in two dimensions as given in [3].

4. Conclusion and Future Work

In this research article, we have presented dynamic inequalities on diamond-α calculus. If we set $\alpha = 1$, then we get delta versions of dynamic inequalities and if we set $\alpha = 0$, then we get nabla versions of dynamic inequalities. Also we get discrete versions of dynamic inequalities, if we put $\mathbb{T} = \mathbb{Z}$ and we get continuous versions of dynamic inequalities, if we put $\mathbb{T} = \mathbb{R}$. Recently various results are developed concerning time scale calculus to produce related dynamic inequalities. It is proved that many classical inequalities such as Radon's Inequality, the weighted power mean inequality, Schlömilch's Inequality, Rogers-Hölder's Inequality and Bernoulli's Inequality are equivalent on time scales as given in [22]. Symmetric versions of Schlömilch's Inequality are also given in [13, 20, 24]. Constantin's Inequality for nabla and diamond-alpha derivative is given in [12]. Some dynamic inequalities using fractional Riemann-Liouville integral on time scales are presented in [21].

It will be interesting to present dynamic inequalities in more dimensions as we have presented here two dimensional versions of Radon's and Cauchy-Schwarz's Inequalities. We can find inequalities using quantum calculus and α, β-symmetric calculus. We can generalize dynamic inequalities using functional generalization. We can also generalize dynamic inequalities of this article using fractional Riemann-Liouville integral and time scales fractional derivative.

References

[1] J. Aczél and E.F. Beckenbach, *On Hölder's Inequality, General inequalities* 2, Proc. Second Internat. Conf. Oberwolfach, (1978), 145-150.
[2] R.P. Agarwal, D. O'Regan and S.H. Saker, *Dynamic inequalities on time scales*, Springer International Publishing, Cham, Switzerland, 2014.
[3] M.R.S. Ammi and D.F.M. Torres, *Hölder's and Hardy's two dimensional diamond-alpha inequalities on time scales*, Ann. Univ. Craiova, Math. Comp. Sci. Series, **37**(1) (2010), 1-11.
[4] D. Anderson, J. Bullock, L. Erbe, A. Peterson and H. Tran, *Nabla dynamic equations on time scales*, Pan-American Mathematical Journal, **13**(1) (2003), 1-48.
[5] E.F. Beckenbach and R. Bellman, *Inequalities*, Springer, Berlin, Göttingen and Heidelberg, 1961.
[6] R. Bellman, *Notes on matrix theory-IV* (An inequality due to Bergström), Amer. Math. Monthly, **62** (1955), 172-173.

[7] H. Bergström, *A triangle inequality for matrices*, Den Elfte Skandinaviske Matematikerkongress, (1949), Trondheim, Johan Grundt Tanums Forlag, Oslo, (1952), 264-267.
[8] M. Bohner and A. Peterson, *Dynamic equations on time scales*, Birkhäuser Boston, Inc., Boston, MA, 2001.
[9] M. Bohner and A. Peterson, *Advances in dynamic equations on time scales*, Birkhäuser Boston, Boston, MA, 2003.
[10] C. Dinu, *Convex functions on time scales*, Annals of the University of Craiova, Math. Comp. Sci. Ser., **35** (2008), 87-96.
[11] D.M. Bătineţu-Giurgiu and O.T. Pop, *A generalization of Radon's Inequality*, Creative Math & Inf., **19**(2) (2010), 116-121.
[12] A.F. Güvenilir, B. Kaymakçalan and N.N. Pelen, *Constantin's inequality for nabla and diamond-alpha derivative*, Journal of Inequalities and Applications, (2015), 1-17.
[13] G.H. Hardy, J.E. Littlewood and G. Pölya, *Inequalities*, 2nd ed, Cambridge, University Press, 1952.
[14] S. Hilger, *Ein Maβkettenkalkül mit Anwendung auf Zentrumsmannigfaltigkeiten*, PhD thesis, Universität Würzburg, 1988.
[15] C.H. Hong and C.C. Yeh, *Rogers-Hölder's Inequality on time scales*, International Journal of Pure and Applied Mathematics, **29**(3) (2006), 289-309.
[16] D.S. Mitrinović, *Analytic inequalities*, Springer-Verlag, Berlin, 1970.
[17] C.P. Niculescu and L.E. Persson, *Convex functions and their applications: A contemporary approach*, CMS Books in Mathematics, Springer, New York, 2006.
[18] U.M. Özkan, M.Z. Sarikaya and H. Yildirim, *Extensions of certain integral inequalities on time scales*, Applied Mathematics Letters, **21** (2008), 993-1000.
[19] J. Radon, *Theorie und Anwendungen der absolut additiven Mengenfunktionen*, Sitzungsber. Acad. Wissen. Wien, **122** (1913), 1295-1438.
[20] M.J.S. Sahir, *Dynamic inequalities for convex functions harmonized on time scales*, Journal of Applied Mathematics and Physics, **5** (2017), 2360-2370.
[21] M.J.S. Sahir, *Fractional dynamic inequalities harmonized on time scales*, Cogent Mathematics & Statistics, **5** (2018), 1-7.
[22] M.J.S. Sahir, *Hybridization of classical inequalities with equivalent dynamic inequalities on time scale calculus*, The Teaching of Mathematics, **21**(1) (2018), 38-52.
[23] Q. Sheng, M. Fadag, J. Henderson and J.M. Davis, *An exploration of combined dynamic derivatives on time scales and their applications*, Nonlinear Anal. Real World Appl., **7** (2006), 395-413.
[24] J.F. Tian and M.H. Ha, *Extensions of Hölder-type inequalities on time scales and their applications*, J. Nonlinear Sci. Appl., **10** (2017), 937-953.

[1]Department of Mathematics,
University of Sargodha, Sub-Campus Bhakkar, Pakistan
& GHSS, 67/ML, Bhakkar, Pakistan
E-mail address: jibrielshahab@gmail.com