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FIGURATIVE INTERPRETATIONS OF BASIC INEQUALITIES FOR CONVEX FUNCTIONS

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Abstract. We give a short survey of basic properties of convex functions and their inequalities referring to a bounded closed interval of real numbers. Given a convex function, its fundamental double inequality is shown in the figurative form. In this review, the Jensen and the Hermite-Hadamard inequalities have a prominent role.

1. Introduction

Let us briefly recall the concepts of affinity and convexity in general. Let $\mathbb{X}$ be a vector space over the field $\mathbb{R}$. A linear combination $\sum_{i=1}^n \lambda_i x_i$ of vectors $x_i \in \mathbb{X}$ and scalars $\lambda_i \in \mathbb{R}$ is said to be affine (convex) if $\sum_{i=1}^n \lambda_i = 1$ (all $\lambda_i \in [0,1]$ and $\sum_{i=1}^n \lambda_i = 1$). A set $\mathcal{S}$ in $\mathbb{X}$ is said to be affine (convex) if it contains all binomial affine (convex) combinations $\lambda_1 x_1 + \lambda_2 x_2$ of all pairs of points $x_1, x_2 \in \mathcal{S}$. Consequently, for each positive integer n the affine (convex) set $\mathcal{S}$ contains all affine (convex) combinations $\sum_{i=1}^n \lambda_i x_i$ of all n-tuples of points $x_1, \dots, x_n \in \mathcal{S}$.

The affine plane (convex polytope) spanned by points $P_1, \dots, P_n \in \mathbb{X}$ is the affine (convex) hull of the set $\{P_1, \dots, P_n\}$. So, the affine plane is the affine set

$$\mathcal{H} = \operatorname{aff}\Big\{P_1, \dots, P_n\Big\} = \Big\{\sum_{i=1}^n \lambda_i P_i : \sum_{i=1}^n \lambda_i = 1\Big\},$$

and the convex polytope is the convex set

$$\mathcal{P} = \operatorname{conv}\Big\{P_1, \dots, P_n\Big\} = \Big\{\sum_{i=1}^n \lambda_i P_i : \lambda_i \in [0,1],\ \sum_{i=1}^n \lambda_i = 1\Big\}.$$

Affine planes are lines in $\mathbb{X} = \mathbb{R}^2$, and lines and planes in $\mathbb{X} = \mathbb{R}^3$. Convex polytopes are bounded closed intervals in $\mathbb{X} = \mathbb{R}$, triangles and other convex polygons in $\mathbb{X} = \mathbb{R}^2$, and tetrahedrons and other convex polyhedrons in $\mathbb{X} = \mathbb{R}^3$.

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If $\mathcal{A}$ is an affine set, then a function $h : \mathcal{A} \to \mathbb{R}$ is said to be affine if the equality

$$h(\lambda_1 x_1 + \lambda_2 x_2) = \lambda_1 h(x_1) + \lambda_2 h(x_2)$$

holds for all binomial affine combinations $\lambda_1 x_1 + \lambda_2 x_2$ of points $x_1, x_2 \in \mathcal{A}$. If $\mathcal{C}$ is a convex set, then a function $f : \mathcal{C} \to \mathbb{R}$ is said to be convex if the inequality

$$f(\lambda_1 x_1 + \lambda_2 x_2) \leq \lambda_1 f(x_1) + \lambda_2 f(x_2)$$

holds for all binomial convex combinations $\lambda_1 x_1 + \lambda_2 x_2$ of points $x_1, x_2 \in \mathcal{C}$.

2. Affinity and convexity concerning the real line

Throughout the paper, we consider a bounded closed interval $[a, b]$ in $\mathbb{R}$ with endpoints $a < b$. Each point $x \in \mathbb{R}$ can be represented as the affine combination

$$x = \alpha a + \beta b \tag{2.1}$$

with coefficients

$$\alpha = \frac{b - x}{b - a}, \quad \beta = \frac{x - a}{b - a}. \tag{2.2}$$

If the point x is in $[a, b]$, then the coefficients α and β are in $[0, 1]$, which indicates that the combination in formula (2.1) is convex.

We recall the fundamental properties of a convex function $f : [a, b] \to \mathbb{R}$. The following lemma refers to the convex function slopes.

Lemma A. *Let $f : [a, b] \to \mathbb{R}$ be a convex function, let $c \in (a, b)$ be an interior point, and let $k \in [f'(c-), f'(c+)]$ be a subderivative of f at c.*

Then the slopes function

$$f_c(x) = \begin{cases} \dfrac{f(x) - f(c)}{x - c} & \text{for} \quad x \neq c \\ k & \text{for} \quad x = c \end{cases} \tag{2.3}$$

is nondecreasing on the interval $[a, b]$.

Figure 1 visualizes the basic features of the observed convex function f relating to the interior point c, and endpoints a and b. The left tangent at c is written by $y = f'(c-)x + l$, and the right tangent at c is written by $y = f'(c+)x + r$.

The convex function f is bounded by two affine functions, the support line

$$h_1(x) = k(x - c) + f(c) \tag{2.4}$$

passing through the point $(c, f(c))$ with the slope k, and the secant line

$$h_2(x) = \frac{b - x}{b - a} f(a) + \frac{x - a}{b - a} f(b) \tag{2.5}$$

passing through the points $(a, f(a))$ and $(b, f(b))$. So, the basic inequality

$$h_1(x) \leq f(x) \leq h_2(x) \tag{2.6}$$

holds for every $x \in [a, b]$. The left-hand side of the inequality in formula (2.6) can be obtained by using separately $f_c(x) \leq k$ for $x < c$, and $f_c(x) \geq k$ for $x > c$. Then it follows that $k(x-c) \leq f_c(x)(x-c) = f(x) - f(c)$ in both cases. The right-hand side of the inequality in formula (2.6) can be obtained by applying the convexity of f to the convex combination in formula (2.1).

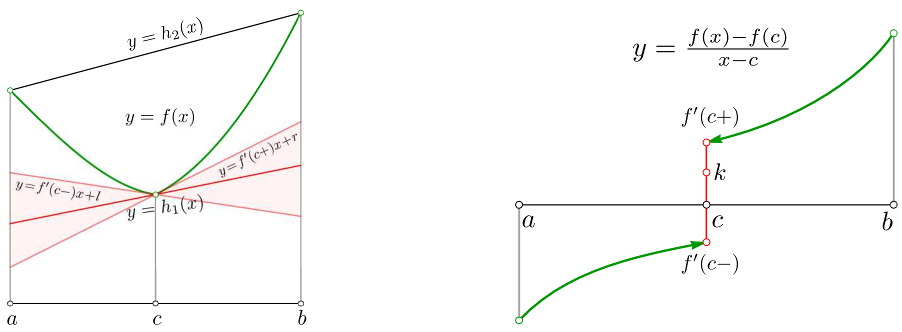

FIGURE 1. Basic analysis of a convex curve

A great source of convex functions are integrals of nondecreasing functions. Let $I = [a, b]$ or $I = [a, \infty)$ be a closed interval, and let $g : I \to \mathbb{R}$ be a nondecreasing function. The function g is almost everywhere continuous, for each $x \in I$ it is bounded on $[a, x]$, and therefore it is integrable on $[a, x]$. Then we have the function $f(x) = \int_a^x g(t)\,dt + r$ which is convex on the interval I for any real constant r. Typical examples of such convex functions are the following:

$$\int_0^x p\,t^{p-1}dt + 1 = x^p \qquad \text{for } p \geq 1 \text{ and } x \geq 0$$

$$\int_a^x p\,t^{p-1}dt + a^p = x^p \qquad \text{for } p < 0 \text{ and } x \geq a > 0$$

$$\int_a^x (-t^{-1})dt - \ln a = -\ln x \qquad \text{for } x \geq a > 0$$

$$\int_a^x e^t dt + e^a = e^x \qquad \text{for } x \geq a > -\infty$$

The integral of the slopes function f_c in equation (2.3) also provides a convex function. More details on the convexity can be found in the books [12], [5] and [15].

3. Inequalities including the convex curve, polygon and secant

The next lemma compares convex combinations of the convex function values. The starting convex combinations of the interval points have the same center.

Lemma B. *Let $c = \sum_{i=1}^n \lambda_i x_i$ be a convex combination of points $x_i \in [a, b]$, and let $c = \alpha a + \beta b$ be the convex combination of endpoints a and b.*

Then each convex function $f : [a, b] \to \mathbb{R}$ satisfies the double inequality

$$f(\alpha a + \beta b) \leq \sum_{i=1}^n \lambda_i f(x_i) \leq \alpha f(a) + \beta f(b). \tag{3.1}$$

Proof. The coefficients α and β can be determined explicitly. Applying formula (2.2) to the point $c \in [a,b]$, we get

$$\alpha = \frac{b-c}{b-a}, \quad \beta = \frac{c-a}{b-a}.$$

Using the points $A=(a,f(a))$, $B=(b,f(b))$ and $P_i=(x_i,f(x_i))$ located on the convex curve $y=f(x)$, we stand out the following three points:

$$F = (c, f(c)) = (c, f(\alpha a+\beta b))$$
$$P = \sum_{i=1}^{n} \lambda_i P_i = \left(\sum_{i=1}^{n} \lambda_i x_i, \sum_{i=1}^{n} \lambda_i f(x_i)\right) = \left(c, \sum_{i=1}^{n} \lambda_i f(x_i)\right)$$
$$C = \alpha A + \beta B = (\alpha a + \beta b, \alpha f(a) + \beta f(b)) = (c, \alpha f(a) + \beta f(b))$$

The point F belongs to the curve $y=f(x)$, the point P belongs to the convex polygon $\mathcal{P}$ spanned by points P_i, and the point C belongs to the secant $\overline{AB}$. The polygon $\mathcal{P}$ is located between the curve $y=f(x)$ and secant $\overline{AB}$ because its sides are curve secants. Since the points F, P and C have the same abscissa $x=c$, the order of their ordinates establishes the double inequality in formula (3.1). □

A figurative explanation of Lemma B is presented in Figure 2. A similar figurative approach was applied in [10].

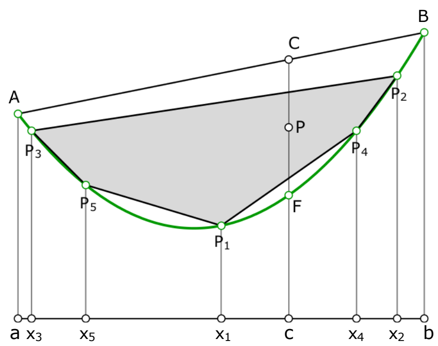

FIGURE 2. The points F, P and C with the same abscissa

Remark 3.1. Let us say something about the strict analytic proof of Lemma B. If $c \in (a,b)$, then the proof is based on the inequality in formula (2.6) by employing the affinity of the support and secant lines. If $c=a$ or $c=b$, then the double inequality in formula (3.1) is reduced to the trivial double equality $f(c)=f(c)=f(c)$.

4. Versions of the Hermite-Hadamard inequality

Upgrading the inequality in formula (3.1) by using the Riemann integral, we can obtain the Hermite-Hadamard inequality (see [2] and [1]).

Corollary 4.1. *Each convex function $f:[a,b]\to\mathbb{R}$ satisfies the double inequality*

$$f\left(\frac{a+b}{2}\right) \le \frac{\int_a^b f(x)\,dx}{b-a} \le \frac{f(a)+f(b)}{2}. \tag{4.1}$$

Proof. Given a positive integer n, we take the intervals

$$I_{ni} = \Big[a + \frac{b-a}{n}(i-1),\, a + \frac{b-a}{n}i\Big]$$

and its midpoints

$$x_{ni} = a + \frac{b-a}{n}\left(i - \frac{1}{2}\right)$$

for every index $i = 1, \ldots, n$. The intervals I_{ni} cover the interval $I = [a, b]$ so that each pair of the adjacent intervals has a common endpoint, and all I_{ni} have the same length $|I_{ni}| = |I|/n = (b-a)/n$.

Using the midpoints x_{ni}, and the coefficients

$$\lambda_{ni} = \frac{1}{n} = \frac{|I_{ni}|}{b-a},$$

we compose the convex combination

$$c_n = \sum_{i=1}^{n} \lambda_{ni} x_{ni} = \frac{a+b}{2}.$$

Applying the inequality in formula (3.1) to the above convex combination, we get

$$f\left(\frac{a+b}{2}\right) \le \sum_{i=1}^{n} \lambda_{ni} f(x_{ni}) \le \frac{f(a)+f(b)}{2}. \tag{4.2}$$

Letting n tend to infinity, the middle member approaches the integral arithmetic mean of f through the limit

$$\begin{aligned} \lim_{n\to\infty} \sum_{i=1}^{n} \lambda_{ni} f(x_{ni}) &= \frac{1}{b-a} \lim_{n\to\infty} \sum_{i=1}^{n} f(x_{ni})|I_{ni}| \\ &= \frac{1}{b-a} \int_a^b f(x)\, dx. \end{aligned}$$

So, the inequality in formula (4.2) approaches the inequality in formula (4.1) as n approaches infinity. ☐

The figurative presentation of the inequality in formula (4.1) is initially Figure 2 with the midpoint $c = (a+b)/2$ and convex polygon $\mathcal{P}_n$ spanned by the points $P_{ni} = (x_{ni}, f(x_{ni}))$ for $i = 1, \ldots, n$. Then letting n tend to infinity, the polygons sequence $(\mathcal{P}_n)_{n=1}^{\infty}$ approaches the convex set $\mathcal{C}$ bounded by $y = f(x)$ and $y = h_2(x)$.

A slightly more general version of the Hermite-Hadamard inequality can be obtained by using the support and secant lines.

Corollary 4.2. *Let $f : [a, b] \to \mathbb{R}$ be a convex function, let $c \in (a, b)$ be an interior point, and let $k \in [f'(c-), f'(c+)]$ be a subderivative of f at c.*

Then we have the double inequality

$$k\left(\frac{a+b}{2} - c\right) + f(c) \le \frac{\int_a^b f(x)\, dx}{b-a} \le \frac{f(a)+f(b)}{2}. \tag{4.3}$$

Proof. Integrating the basic inequality in formula (2.6) over the interval $[a, b]$, and dividing by $b - a$, we get

$$\frac{\int_a^b h_1(x)\,dx}{b-a} \le \frac{\int_a^b f(x)\,dx}{b-a} \le \frac{\int_a^b h_2(x)\,dx}{b-a}. \tag{4.4}$$

Applying the affinity of h_1 and h_2 (trapeze area formula), and using the equations in formulae (2.4) and (2.5), we gain

$$\frac{\int_a^b h_1(x)\,dx}{b-a} = \frac{h_1(a)+h_1(b)}{2} = k\left(\frac{a+b}{2} - c\right) + f(c)$$

and

$$\frac{\int_a^b h_2(x)\,dx}{b-a} = \frac{h_2(a)+h_2(b)}{2} = \frac{f(a)+f(b)}{2}.$$

Resetting the inequality in formula (4.4) by the above calculations, we reach the inequality in formula (4.3). ☐

Formula (4.3) is reduced to formula (4.1) if we put $c = (a+b)/2$. Does there exist a point different from c that improves the left-hand side of the Hermite-Hadamard inequality? There does not exist such point because the following is true.

As for the first member in formula (4.3), the inequality

$$k\left(\frac{a+b}{2} - c\right) + f(c) \le f\left(\frac{a+b}{2}\right) \tag{4.5}$$

holds for every $c \in (a, b)$. If $c = (a+b)/2$, we have the equality. If $c \neq (a+b)/2$, we use the slopes function f_c in formula (2.3) as follows. If $c < (a+b)/2$, then the inequality

$$k \le f_c\left(\frac{a+b}{2}\right) = \frac{f\left(\frac{a+b}{2}\right) - f(c)}{\frac{a+b}{2} - c}$$

is valid, and if $c > (a+b)/2$, then the reverse inequality is valid. In both cases, multiplying by the denominator, and moving $f(c)$ over to the left side, it follows the inequality in formula (4.5).

To realize many particular inequalities, the derivative form of the Hermite-Hadamard inequality is probably the most convenient. It is as follows.

Corollary 4.3. *Let $f : [a, b] \to \mathbb{R}$ be a function such that its derivative f' exists on the open interval (a, b) as a bounded convex function.*

Then we have the double inequality

$$f'\left(\frac{a+b}{2}\right) \le \frac{f(b)-f(a)}{b-a} \le \frac{f'(a+)+f'(b-)}{2}. \tag{4.6}$$

Proof. Since f' is convex, it is continuous. Thus f' is bounded and continuous, and therefore $f'(a+)$ and $f'(b-)$ exist as real numbers. Putting $f'(a) = f'(a+)$ and $f'(b) = f'(b-)$, we can use f' as the continuous convex function on $[a, b]$. Substituting f with f' in formula (4.1), and employing the Leibniz-Newton formula through the equality $\int_a^b f'(x)\,dx = f(b) - f(a)$, we get formula (4.6). ☐

The simplest way to obtain the Hermite-Hadamard inequality for a convex function $f : [a, b] \to \mathbb{R}$ is as follows.

Applying the convexity of f to the convex combination

$$\frac{a+b}{2} = \frac{1}{2}\big((1-t)a+tb\big) + \frac{1}{2}\big(ta+(1-t)b\big),$$

we get the inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{2}f\big((1-t)a+tb\big) + \frac{1}{2}f\big(ta+(1-t)b\big)$$

which holds for every $t \in [0, 1]$. Integrating the above inequality by t over the interval $[0, 1]$ via substitutions $x = (1-t)a+tb$ and $x = ta+(1-t)b$, we reach the left-hand side

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{2}\frac{\int_a^b f(x)\,dx}{b-a} + \frac{1}{2}\frac{\int_b^a f(x)\,dx}{a-b} = \frac{\int_a^b f(x)\,dx}{b-a}.$$

Applying the similar procedure to the convexity definition inequality

$$f\big((1-t)a+tb\big) \leq (1-t)f(a) + tf(b),$$

we reach the right-hand side

$$\frac{\int_a^b f(x)\,dx}{b-a} \leq \frac{f(a)+f(b)}{2}.$$

An interesting historical story about the Hermite-Hadamard inequality can be read in [7]. Improvements of this inequality can be seen in [11]. As far as the wide application of this inequality, one can be informed in [8], [13] and [14].

5. Versions of the Jensen inequality

The discrete form of the Jensen inequality (see [3]) can be extended to the right. The extension is a direct consequence of Lemma B.

Corollary 5.1. *Let $c = \sum_{i=1}^n \lambda_i x_i$ be a convex combination of points $x_i \in [a, b]$.*

Then each convex function $f : [a, b] \to \mathbb{R}$ satisfies the double inequality

$$f\left(\sum_{i=1}^n \lambda_i x_i\right) \leq \sum_{i=1}^n \lambda_i f(x_i) \leq \frac{b-c}{b-a}f(a) + \frac{c-a}{b-a}f(b). \tag{5.1}$$

Relying on the proof of Corollary 4.1, the discrete inequality in formula (5.1) can be upgraded to the integral form. In this way, we obtain the integral form of the Jensen inequality (see [4]) extended to the right side.

Corollary 5.2. *Let $g : [u, v] \to \mathbb{R}$ be a bounded integrable function, and let*

$$c = \frac{1}{v-u}\int_u^v g(x)\,dx$$

be its integral arithmetic mean. Let $[a, b]$ be an interval containing the image of g.

Then each convex function $f : [a, b] \to \mathbb{R}$ satisfies the double inequality

$$f\left(\frac{\int_u^v g(x)\,dx}{v-u}\right) \leq \frac{\int_u^v f(g(x))\,dx}{v-u} \leq \frac{b-c}{b-a}f(a) + \frac{c-a}{b-a}f(b). \tag{5.2}$$

Proof. Let $I=[u,v]$, and let $|I_{ni}|=|I|/n=(v-u)/n$. Fitting the function g into the proof of Corollary 4.1, we will rely on the convex combination

$$c_n=\sum_{i=1}^{n}\lambda_{ni}\,g(x_{ni}),$$

where $x_{ni}=a+(i-1/2)(v-u)/n$ and $\lambda_{ni}=1/n=|I_{ni}|(v-u)$, and at the conclusion of the proof, we will use the fact that $\lim_{n\to\infty}c_n=c$.

Applying formula (5.1) to the convex combination c_n, we get

$$f\Big(\sum_{i=1}^{n}\lambda_{ni}\,g(x_{ni})\Big)\le\sum_{i=1}^{n}\lambda_{ni}f(g(x_{ni}))\le\frac{b-c_n}{b-a}f(a)+\frac{c_n-a}{b-a}f(b).$$

If $c\in(a,b)$, then letting n tend to infinity, the above inequality tends to the inequality in formula (5.2). As for the first member, we can employ the continuity of f on (a,b), and so get

$$\lim_{n\to\infty}f(c_n)=f\big(\lim_{n\to\infty}c_n\big)=f(c). \tag{5.3}$$

If $c=a$ or $c=b$, then the function g is almost everywhere equal to c. In this case, the trivial double inequality $f(c)\le f(c)\le f(c)$ represents formula (5.2). $\square$

Remark 5.1. We point out two characteristic details regarding the limit of a convex function $f:[a,b]\to\mathbb{R}$. If $(c_n)_{n=1}^{\infty}$ is a convergent sequence of points $c_n\in[a,b]$, then it follows the inequality

$$\lim_{n\to\infty}f(c_n)\le f\big(\lim_{n\to\infty}c_n\big).$$

Namely, if f is continuous and if $\lim_{n\to\infty}c_n=c$ belonging to $[a,b]$, then the equality in formula (5.3) applies, but if f is discontinued at the endpoint a and if $\lim_{n\to\infty}c_n=a$, then

$$f(a+)=\lim_{n\to\infty}f(c_n)<f\big(\lim_{n\to\infty}c_n\big)=f(a).$$

Extensions of the Jensen inequality to affine combinations were done in [9]. The converse form of the Jensen inequality, as well as its application to convex and related functions were presented in [6].

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