Turkish J. Ineq., 2 (1) (2018), Pages 1 –20 .

Turkish Journal of
INEQUALITIES

Available online at www.tjinequality.com

FRACTIONAL HERMITE-HADAMARD-FEJER TYPE INEQUALITIES FOR GA-CONVEX FUNCTIONS

MEHMET KUNT[1] AND İMDAT İŞCAN[2]

ABSTRACT. In this paper, a fractional Hermite-Hadamard type inequality and two different fractional Hermite-Hadamard-Fejer type inequalities for GA-convex functions are proved. Also, two identity for differentiable functions are obtained. By using this two identity, some trapezoid and midpoint type errors estimations for GA-convex functions in fractional integral forms are established.

1. INTRODUCTION

Definition 1.1. [6, 7]. A function $f : I \subseteq (0,\infty) \to \mathbb{R}$ is a GA-convex (geometric-arithmetically convex), if

$$f(x^t y^{1-t}) \leq tf(x) + (1-t)f(y)$$

for all $x, y \in I$ and $t \in [0,1]$.

In recent decades, many authors have studied in errors estimations for Hermite-Hadamard's inequalities for GA-convex functions, see [2, 5] and references therein.

Definition 1.2. [4]. Let $f \in L[a,b]$. The right-hand side and left-hand side Hadamard fractional integrals $J^{\alpha}_{a+}f$ and $J^{\alpha}_{b-}f$ of order $\alpha > 0$ with $b > a \geq 0$ are defined by

$$J^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x \left(\ln\frac{x}{t}\right)^{\alpha-1} f(t)\frac{dt}{t},\ x > a$$

and

$$J^{\alpha}_{b-}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b \left(\ln\frac{t}{x}\right)^{\alpha-1} f(t)\frac{dt}{t},\ x < b$$

respectively, where $\Gamma(\alpha)$ is the Gamma function defined by $\Gamma(\alpha) = \int_0^{\infty} e^{-t}t^{\alpha-1}dt$.

Recently, some Hermite-Hadamard inequalities involving fractional integrals have been obtained for different classes of functions; see [1–3, 9, 10] and references therein.

Key words and phrases. Hermite-Hadamard inequality, Hermite-Hadamard-Fejer inequality, Hadamard fractional integral, GA-convex functions.

2010 *Mathematics Subject Classification.* 26A51, 26A33, 26D10.

Received: 15/09/2017

Accepted: 02/01/2018.

Lemma 1.1. [8,10]. *For* $0<\alpha\leq 1$ *and* $0\leq a<b$, *we have* $|a^{\alpha}-b^{\alpha}|\leq (b-a)^{\alpha}$.

In this paper, we will prove a fractional Hermite-Hadamard type inequalities and two different fractional Hermite-Hadamard-Fejer type inequalities for GA-convex functions. Also, we will obtain two identity for differentiable functions. By using this two identity, we will establish some trapezoid and midpoint type errors estimations for GA-convex functions in fractional integral forms.

2. Main Results

Throughout this section, let $\|g\|_{\infty}=\sup_{t\in[a,b]}|g(t)|$, for the continuous function $g:[a,b]\rightarrow\mathbb{R}$.

Lemma 2.1. *If* $g:[a,b]\subset(0,\infty)\rightarrow\mathbb{R}$ *is integrable and geometrically symmetric with respect to* $\sqrt{ab}$ *(i.e.* $g\left(\frac{ab}{x}\right)=g(x)$ *holds for all* $x\in[a,b]$*) with* $a<b$ *and* $\alpha>0$, *then*

$$J_{a+}^{\alpha}g(b)=J_{b-}^{\alpha}g(a)=\frac{1}{2}\left[J_{a+}^{\alpha}g(b)+J_{b-}^{\alpha}g(a)\right].$$

Proof. Using the geometrically symmetricity of g with respect to $\sqrt{ab}$, we have $g(ab/x)=g(x)$, for all $x\in[a,b]$. In the following integral, if we setting $x=ab/t$ and $dx=-(ab/t^2)\,dt$, we have

$$\begin{aligned} J_{a+}^{\alpha}g(b) &= \frac{1}{\Gamma(\alpha)}\int_a^b\left(\ln\frac{b}{x}\right)^{\alpha-1}g(x)\frac{dx}{x}=\frac{1}{\Gamma(\alpha)}\int_a^b\left(\ln\frac{t}{a}\right)^{\alpha-1}g\left(\frac{ab}{t}\right)\frac{dt}{t}\\ &= \frac{1}{\Gamma(\alpha)}\int_a^b\left(\ln\frac{t}{a}\right)^{\alpha-1}g(t)\,\frac{dt}{t}=J_{b-}^{\alpha}g(a). \end{aligned}$$

□

Theorem 2.1. *Let* $f:I\subset(0,\infty)\rightarrow\mathbb{R}$ *be a* GA*-convex function such that* $f\in L[a,b]$ *where* $a,b\in I$ *with* $a<b$ *and* $\alpha>0$. *If* $g:[a,b]\rightarrow\mathbb{R}$ *is nonnegative, integrable and geometrically symmetric with respect to* $\sqrt{ab}$, *then the following inequalities for fractional integrals holds:*

$$\begin{aligned} f\left(\sqrt{ab}\right)\left[J_{a+}^{\alpha}g(b)+J_{b-}^{\alpha}g(a)\right] &\leq \left[J_{a+}^{\alpha}(fg)(b)+J_{b-}^{\alpha}(fg)(a)\right] \qquad (2.1)\\ &\leq \frac{f(a)+f(b)}{2}\left[J_{a+}^{\alpha}g(b)+J_{b-}^{\alpha}g(a)\right]. \end{aligned}$$

Proof. Using GA-convexity of the function f on $[a,b]$, we have

$$f\left(\sqrt{ab}\right)=f\left(\sqrt{a^tb^{1-t}.a^{1-t}b^t}\right)\leq\frac{f\left(a^tb^{1-t}\right)+f\left(a^{1-t}b^t\right)}{2}, \qquad (2.2)$$

for all $t\in[0,1]$.

Multiplying both sides of (2.2) by $2t^{\alpha-1}g\left(a^{1-t}b^t\right)$, and integrating the resulting inequality with respect to t over $[0,1]$, we have

$$\begin{aligned} 2f\left(\sqrt{ab}\right)\int_0^1 t^{\alpha-1}g\left(a^{1-t}b^t\right)dt &\leq \int_0^1 t^{\alpha-1}\left[f\left(a^tb^{1-t}\right)+f\left(a^{1-t}b^t\right)\right]g\left(a^{1-t}b^t\right)dt\\ &= \int_0^1 t^{\alpha-1}f\left(a^tb^{1-t}\right)g\left(a^{1-t}b^t\right)dt+\int_0^1 t^{\alpha-1}f\left(a^{1-t}b^t\right)g\left(a^{1-t}b^t\right)dt. \end{aligned}$$

Setting $x = a^{1-t}b^t$, and $dx = a^{1-t}b^t \ln\left(\frac{b}{a}\right) dt$, we have

$$\begin{aligned}
&\frac{2}{\left(\ln \frac{b}{a}\right)^{\alpha}} f\left(\sqrt{ab}\right) \int_a^b \left(\ln \frac{x}{a}\right)^{\alpha-1} g(x) \frac{dx}{x} \\
&\leq \frac{1}{\left(\ln \frac{b}{a}\right)^{\alpha}} \left\{ \int_a^b \left(\ln \frac{x}{a}\right)^{\alpha-1} f\left(\frac{ab}{x}\right) g(x) \frac{dx}{x} + \int_a^b \left(\ln \frac{x}{a}\right)^{\alpha-1} f(x) g(x) \frac{dx}{x} \right\} \\
&= \frac{1}{\left(\ln \frac{b}{a}\right)^{\alpha}} \left\{ \int_a^b \left(\ln \frac{b}{x}\right)^{\alpha-1} f(x) g\left(\frac{ab}{x}\right) \frac{dx}{x} + \int_a^b \left(\ln \frac{x}{a}\right)^{\alpha-1} f(x) g(x) \frac{dx}{x} \right\} \\
&= \frac{1}{\left(\ln \frac{b}{a}\right)^{\alpha}} \left\{ \int_a^b \left(\ln \frac{b}{x}\right)^{\alpha-1} f(x) g(x) \frac{dx}{x} + \int_a^b \left(\ln \frac{x}{a}\right)^{\alpha-1} f(x) g(x) \frac{dx}{x} \right\}.
\end{aligned}$$

Using Lemma 2.1, we have

$$\frac{\Gamma(\alpha)}{\left(\ln \frac{b}{a}\right)^{\alpha}} f\left(\sqrt{ab}\right) \left[J_{a+}^{\alpha} g(b) + J_{b-}^{\alpha} g(a)\right] \leq \frac{\Gamma(\alpha)}{\left(\ln \frac{b}{a}\right)^{\alpha}} \left[J_{a+}^{\alpha}(fg)(b) + J_{b-}^{\alpha}(fg)(a)\right],$$

which is the proof of the first inequality in (2.1).

For the proof of the second inequality in (2.1), we first note that if f is a GA-convex function, then for all $t \in [0,1]$, it yields

$$f\left(a^t b^{1-t}\right) + f\left(a^{1-t} b^t\right) \leq f(a) + f(b). \tag{2.3}$$

Then multiplying both sides of (2.3) by $t^{\alpha-1} g\left(a^{1-t}b^t\right)$ and integrating the resulting inequality with respect to t over $[0,1]$, we have

$$\begin{aligned}
&\int_0^1 t^{\alpha-1} f\left(a^t b^{1-t}\right) g\left(a^{1-t} b^t\right) dt + \int_0^1 t^{\alpha-1} f\left(a^{1-t} b^t\right) g\left(a^{1-t} b^t\right) dt \\
&\leq \left[f(a) + f(b)\right] \int_0^1 t^{\alpha-1} g\left(a^{1-t} b^t\right) dt.
\end{aligned}$$

Using Lemma 2.1, we have

$$\frac{\Gamma(\alpha)}{\left(\ln \frac{b}{a}\right)^{\alpha}} \left[J_{a+}^{\alpha}(fg)(b) + J_{b-}^{\alpha}(fg)(a)\right] \leq \frac{\Gamma(\alpha)}{\left(\ln \frac{b}{a}\right)^{\alpha}} \left(\frac{f(a)+f(b)}{2}\right) \left[J_{a+}^{\alpha} g(b) + J_{b-}^{\alpha} g(a)\right],$$

which is the proof of the second inequality in (2.1).The proof is completed. □

Remark 2.1. In Theorem 2.1,

(1) if one takes $\alpha = 1$, one recaptures the inequality [5, Theorem 6],
(2) if one takes $g(x) = 1$, one recaptures the inequality [2, Theorem 2.1].

Lemma 2.2. *Let $f : I \subset (0,\infty) \to \mathbb{R}$ be a differentiable mapping on I° such that $f \in L[a,b]$ where $a, b \in I$ with $a < b$ and $\alpha > 0$. If $g : [a,b] \to \mathbb{R}$ is integrable and geometrically symmetric with respect to $\sqrt{ab}$ then the following equality via fractional integrals holds:*

$$\left(\frac{f(a)+f(b)}{2}\right) \left[J_{a+}^{\alpha} g(b) + J_{b-}^{\alpha} g(a)\right] - \left[J_{a+}^{\alpha}(fg)(b) + J_{b-}^{\alpha}(fg)(a)\right] \tag{2.4}$$

$$= \frac{1}{\Gamma(\alpha)} \int_a^b \left[\int_a^t \left(\ln \frac{b}{s} \right)^{\alpha-1} g(s) \frac{ds}{s} - \int_t^b \left(\ln \frac{s}{a} \right)^{\alpha-1} g(s) \frac{ds}{s} \right] f'(t) dt.$$

Proof. It suffices to note that

$$I = \int_a^b \left[\int_a^t \left(\ln \frac{b}{s} \right)^{\alpha-1} g(s) \frac{ds}{s} - \int_t^b \left(\ln \frac{s}{a} \right)^{\alpha-1} g(s) \frac{ds}{s} \right] f'(t) dt \tag{2.5}$$

$$= \int_a^b \left(\int_a^t \left(\ln \frac{b}{s} \right)^{\alpha-1} g(s) \frac{ds}{s} \right) f'(t) dt + \int_a^b \left(- \int_t^b \left(\ln \frac{s}{a} \right)^{\alpha-1} g(s) \frac{ds}{s} \right) f'(t) dt = I_1 + I_2.$$

By integration by parts and using Lemma 2.1, we have

$$\begin{aligned} I_1 &= \left(\int_a^t \left(\ln \frac{b}{s} \right)^{\alpha-1} g(s) \frac{ds}{s} \right) f(t) \Big|_a^b - \int_a^b \left(\ln \frac{b}{t} \right)^{\alpha-1} g(t) f(t) \frac{dt}{t} \\ &= \left(\int_a^b \left(\ln \frac{b}{s} \right)^{\alpha-1} g(s) \frac{ds}{s} \right) f(b) - \int_a^b \left(\ln \frac{b}{t} \right)^{\alpha-1} (fg)(t) \frac{dt}{t} \\ &= \Gamma(\alpha) \left[f(b) J_{a+}^{\alpha} g(b) - J_{a+}^{\alpha} (fg)(b) \right] \\ &= \Gamma(\alpha) \left[\frac{f(b)}{2} \left[J_{a+}^{\alpha} g(b) + J_{b-}^{\alpha} g(a) \right] - J_{a+}^{\alpha} (fg)(b) \right], \end{aligned} \tag{2.6}$$

and similarly

$$\begin{aligned} I_2 &= \left(- \int_t^b \left(\ln \frac{s}{a} \right)^{\alpha-1} g(s) \frac{ds}{s} \right) f(t) \Big|_a^b - \int_a^b \left(\ln \frac{t}{a} \right)^{\alpha-1} g(t) f(t) \frac{dt}{t} \\ &= \left(\int_a^b \left(\ln \frac{s}{a} \right)^{\alpha-1} g(s) \frac{ds}{s} \right) f(a) - \int_a^b \left(\ln \frac{t}{a} \right)^{\alpha-1} (fg)(t) \frac{dt}{t} \\ &= \Gamma(\alpha) \left[\frac{f(a)}{2} \left[J_{a+}^{\alpha} g(b) + J_{b-}^{\alpha} g(a) \right] - J_{b-}^{\alpha} (fg)(a) \right]. \end{aligned} \tag{2.7}$$

Using (2.6) and (2.7) in (2.5), than multiplying the both sides of the resulting inequality by $(\Gamma(\alpha))^{-1}$, we have (2.4). This completes the proof. □

Theorem 2.2. *Let $f : I \subset (0, \infty) \to \mathbb{R}$ be a differentiable mapping on I° such that $f \in L[a, b]$ where $a, b \in I$ with $a < b$ and $\alpha > 0$. If $|f'|$ is GA-convex on $[a, b]$ and $g : [a, b] \to \mathbb{R}$ is continuous and geometrically symmetric with respect to $\sqrt{ab}$, then the following inequality via fractional integrals holds:*

$$\begin{aligned} &\left| \left(\frac{f(a) + f(b)}{2} \right) \left[J_{a+}^{\alpha} g(b) + J_{b-}^{\alpha} g(a) \right] - \left[J_{a+}^{\alpha} (fg)(b) + J_{b-}^{\alpha} (fg)(a) \right] \right| \\ &\leq \frac{\|g\|_{\infty} \ln^{\alpha+1} \left(\frac{b}{a} \right)}{\Gamma(\alpha+1)} \left[C_1(\alpha) \left| f'(a) \right| + C_2(\alpha) \left| f'(b) \right| \right], \end{aligned} \tag{2.8}$$

where

$$C_1(\alpha) = \int_0^{\frac{1}{2}} \left[(1-u)^{\alpha} - u^{\alpha} \right] \left[(1-u) a^{1-u} b^u + u a^u b^{1-u} \right] du,$$

$$C_2(\alpha) = \int_0^{\frac{1}{2}} \left[(1-u)^{\alpha} - u^{\alpha} \right] \left[u a^{1-u} b^u + (1-u) a^u b^{1-u} \right] du.$$

Proof. Using Lemma 2.2, we have

$$\left|\left(\frac{f(a)+f(b)}{2}\right)\left[J^{\alpha}_{a+}g(b)+J^{\alpha}_{b-}g(a)\right]-\left[J^{\alpha}_{a+}(fg)(b)+J^{\alpha}_{b-}(fg)(a)\right]\right|$$
$$\leq \frac{1}{\Gamma(\alpha)}\int_a^b\left|\int_a^t\left(\ln\frac{b}{s}\right)^{\alpha-1}g(s)\frac{ds}{s}-\int_t^b\left(\ln\frac{s}{a}\right)^{\alpha-1}g(s)\frac{ds}{s}\right|\left|f'(t)\right|dt.$$

Setting $t=a^{1-u}b^u$ and $dt=a^{1-u}b^u\ln\left(\frac{b}{a}\right)du$, we have

$$\left|\left(\frac{f(a)+f(b)}{2}\right)\left[J^{\alpha}_{a+}g(b)+J^{\alpha}_{b-}g(a)\right]-\left[J^{\alpha}_{a+}(fg)(b)+J^{\alpha}_{b-}(fg)(a)\right]\right| \tag{2.9}$$
$$\leq \frac{1}{\Gamma(\alpha)}\int_0^1\left|\begin{array}{c}\int_a^{a^{1-u}b^u}\left(\ln\frac{b}{s}\right)^{\alpha-1}g(s)\frac{ds}{s}\\ -\int_{a^{1-u}b^u}^b\left(\ln\frac{s}{a}\right)^{\alpha-1}g(s)\frac{ds}{s}\end{array}\right|\left|f'(a^{1-u}b^u)\right|a^{1-u}b^u\ln\left(\frac{b}{a}\right)du.$$

Using the geometrically symmetricity of g, we have

$$\int_{a^{1-u}b^u}^b\left(\ln\frac{s}{a}\right)^{\alpha-1}g(s)\frac{ds}{s}=\int_a^{a^ub^{1-u}}\left(\ln\frac{b}{s}\right)^{\alpha-1}g(\frac{ab}{s})\frac{ds}{s}=\int_a^{a^ub^{1-u}}\left(\ln\frac{b}{s}\right)^{\alpha-1}g(s)\frac{ds}{s}.$$

Using this equality, we have

$$\left|\int_a^{a^{1-u}b^u}\left(\ln\frac{b}{s}\right)^{\alpha-1}g(s)\frac{ds}{s}-\int_{a^{1-u}b^u}^b\left(\ln\frac{s}{a}\right)^{\alpha-1}g(s)\frac{ds}{s}\right|=\left|\int_{a^{1-u}b^u}^{a^ub^{1-u}}\left(\ln\frac{b}{s}\right)^{\alpha-1}g(s)\frac{ds}{s}\right| \tag{2.10}$$
$$\leq\left\{\begin{array}{ll}\int_{a^{1-u}b^u}^{a^ub^{1-u}}\left(\ln\frac{b}{s}\right)^{\alpha-1}|g(s)|\frac{ds}{s} & u\in\left[0,\frac{1}{2}\right]\\ \int_{a^ub^{1-u}}^{a^{1-u}b^u}\left(\ln\frac{b}{s}\right)^{\alpha-1}|g(s)|\frac{ds}{s} & u\in\left[\frac{1}{2},1\right]\end{array}\right.$$
$$\leq\|g\|_\infty\left\{\begin{array}{ll}\int_{a^{1-u}b^u}^{a^ub^{1-u}}\left(\ln\frac{b}{s}\right)^{\alpha-1}\frac{ds}{s} & u\in\left[0,\frac{1}{2}\right]\\ \int_{a^ub^{1-u}}^{a^{1-u}b^u}\left(\ln\frac{b}{s}\right)^{\alpha-1}\frac{ds}{s} & u\in\left[\frac{1}{2},1\right]\end{array}\right.$$
$$=\|g\|_\infty\frac{\left(\ln\frac{b}{a}\right)^\alpha}{\alpha}\left\{\begin{array}{ll}(1-u)^\alpha-u^\alpha & u\in\left[0,\frac{1}{2}\right]\\ u^\alpha-(1-u)^\alpha & u\in\left[\frac{1}{2},1\right]\end{array}\right..$$

On the other hand, using GA-convexity of $|f'|$ on $[a,b]$, we have

$$\left|f'(a^{1-u}b^u)\right|\leq(1-u)\left|f'(a)\right|+u\left|f'(b)\right|, \tag{2.11}$$

for all $u\in[0,1]$. A combination of (2.9), (2.10) and (2.11), we have

$$\left|\left(\frac{f(a)+f(b)}{2}\right)\left[J^{\alpha}_{a+}g(b)+J^{\alpha}_{b-}g(a)\right]-\left[J^{\alpha}_{a+}(fg)(b)+J^{\alpha}_{b-}(fg)(a)\right]\right| \tag{2.12}$$
$$\leq \frac{1}{\Gamma(\alpha)}\int_0^1\left|\begin{array}{c}\int_a^{a^{1-u}b^u}\left(\ln\frac{b}{s}\right)^{\alpha-1}g(s)\frac{ds}{s}\\ -\int_{a^{1-u}b^u}^b\left(\ln\frac{s}{a}\right)^{\alpha-1}g(s)\frac{ds}{s}\end{array}\right|\left|f'(a^{1-u}b^u)\right|a^{1-u}b^u\ln\left(\frac{b}{a}\right)du.$$
$$\leq\frac{1}{\Gamma(\alpha)}\int_0^{\frac{1}{2}}\left(\|g\|_\infty\frac{\left(\ln\frac{b}{a}\right)^\alpha}{\alpha}\left[(1-u)^\alpha-u^\alpha\right]\right)\left((1-u)\left|f'(a)\right|+u\left|f'(b)\right|\right)a^{1-u}b^u\ln\left(\frac{b}{a}\right)du$$

$$+\frac{1}{\Gamma(\alpha)}\int_{\frac{1}{2}}^{1}\left(\|g\|_{\infty}\frac{\left(\ln\frac{b}{a}\right)^{\alpha}}{\alpha}\left[u^{\alpha}-(1-u)^{\alpha}\right]\right)\left((1-u)\left|f'(a)\right|+u\left|f'(b)\right|\right)a^{1-u}b^{u}\ln\left(\frac{b}{a}\right)du$$

$$\leq\frac{\ln^{\alpha+1}\left(\frac{b}{a}\right)\|g\|_{\infty}}{\Gamma(\alpha+1)}\left\{\int_{0}^{\frac{1}{2}}\left[(1-u)^{\alpha}-u^{\alpha}\right]\left((1-u)\left|f'(a)\right|+u\left|f'(b)\right|\right)a^{1-u}b^{u}du\right.$$

$$\left.+\int_{\frac{1}{2}}^{1}\left[u^{\alpha}-(1-u)^{\alpha}\right]\left((1-u)\left|f'(a)\right|+u\left|f'(b)\right|\right)a^{1-u}b^{u}du\right\}.$$

An easy calculation, we have

$$\int_{\frac{1}{2}}^{1}\left[u^{\alpha}-(1-u)^{\alpha}\right](1-u)\,a^{1-u}b^{u}du=\int_{0}^{\frac{1}{2}}\left[(1-u)^{\alpha}-u^{\alpha}\right]ua^{u}b^{1-u}du, \tag{2.13}$$

and

$$\int_{\frac{1}{2}}^{1}\left[u^{\alpha}-(1-u)^{\alpha}\right]ua^{1-u}b^{u}du=\int_{0}^{\frac{1}{2}}\left[(1-u)^{\alpha}-u^{\alpha}\right](1-u)\,a^{u}b^{1-u}du. \tag{2.14}$$

Finally, if we use (2.13) and (2.14) in (2.12), we obtain

$$\left|\left(\frac{f(a)+f(b)}{2}\right)\left[J_{a+}^{\alpha}g(b)+J_{b-}^{\alpha}g(a)\right]-\left[J_{a+}^{\alpha}(fg)(b)+J_{b-}^{\alpha}(fg)(a)\right]\right|$$

$$\leq\frac{\ln^{\alpha+1}\left(\frac{b}{a}\right)\|g\|_{\infty}}{\Gamma(\alpha+1)}\left\{\int_{0}^{\frac{1}{2}}\left[(1-u)^{\alpha}-u^{\alpha}\right]\left[(1-u)\,a^{1-u}b^{u}+ua^{u}b^{1-u}\right]du\left|f'(a)\right|\right.$$

$$\left.+\int_{0}^{\frac{1}{2}}\left[(1-u)^{\alpha}-u^{\alpha}\right]\left[ua^{1-u}b^{u}+(1-u)\,a^{u}b^{1-u}\right]du\left|f'(b)\right|\right\}.$$

This completes the proof. □

Corollary 2.1. *In Theorem 2.2; if one takes $\alpha=1$, $g(x)=1$ and $\alpha=1=g(x)$ respectively, one has the following inequalities:*

$$\left|\frac{f(a)+f(b)}{2}\int_{a}^{b}\frac{g(x)}{x}dx-\int_{a}^{b}\frac{f(x)g(x)}{x}dx\right| \tag{2.15}$$

$$\leq\ \|g\|_{\infty}\frac{\ln^{2}\left(\frac{b}{a}\right)}{2}\left[C_{1}(1)\left|f'(a)\right|+C_{2}(1)\left|f'(b)\right|\right],$$

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2\left(\ln\frac{b}{a}\right)^{\alpha}}\left[J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)\right]\right|\leq\frac{\ln\left(\frac{b}{a}\right)}{2}\left[C_{1}(\alpha)\left|f'(a)\right|+C_{2}(\alpha)\left|f'(b)\right|\right], \tag{2.16}$$

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{\ln\frac{b}{a}}\int_{a}^{b}\frac{f(x)}{x}dx\right|\leq\frac{\ln\left(\frac{b}{a}\right)}{2}\left[C_{1}(1)\left|f'(a)\right|+C_{2}(1)\left|f'(b)\right|\right]. \tag{2.17}$$

Theorem 2.3. *Let $f : I \subset (0,\infty) \to \mathbb{R}$ be a differentiable mapping on I° such that $f \in L[a,b]$ where $a,b \in I$ with $a < b$ and $\alpha > 0$. If $|f'|^q, q \geq 1$, is GA-convex on $[a,b]$ and $g : [a,b] \to \mathbb{R}$ is continuous and geometrically symmetric with respect to $\sqrt{ab}$, then the following inequalities via fractional integrals holds:*

$$\left|\left(\frac{f(a)+f(b)}{2}\right)\left[J_{a+}^{\alpha}g(b)+J_{b-}^{\alpha}g(a)\right]-\left[J_{a+}^{\alpha}(fg)(b)+J_{b-}^{\alpha}(fg)(a)\right]\right| \tag{2.18}$$
$$\leq \frac{\|g\|_\infty \ln^{\alpha+1}\left(\frac{b}{a}\right)}{\Gamma(\alpha+1)}\left[\left(1-\frac{1}{2^\alpha}\right)\left(\frac{2}{\alpha+1}\right)\right]^{1-\frac{1}{q}}\left[C_3(\alpha)\left|f'(a)\right|^q+C_4(\alpha)\left|f'(b)\right|^q\right]^{\frac{1}{q}},$$

where

$$C_3(\alpha)=\int_0^{\frac{1}{2}}\left[(1-u)^\alpha-u^\alpha\right]\left[(1-u)\left(a^{1-u}b^u\right)^q+u\left(a^u b^{1-u}\right)^q\right]du,$$
$$C_4(\alpha)=\int_0^{\frac{1}{2}}\left[(1-u)^\alpha-u^\alpha\right]\left[u\left(a^{1-u}b^u\right)^q+(1-u)\left(a^u b^{1-u}\right)^q\right]du.$$

Proof. Similarly the proof of Theorem 2.2, using Lemma 2.2, (2.9), (2.10), power mean inequality and GA-convexity of $|f'|^q$, we have

$$\left|\left(\frac{f(a)+f(b)}{2}\right)\left[J_{a+}^{\alpha}g(b)+J_{b-}^{\alpha}g(a)\right]-\left[J_{a+}^{\alpha}(fg)(b)+J_{b-}^{\alpha}(fg)(a)\right]\right|$$
$$\leq \frac{1}{\Gamma(\alpha)}\int_0^1\left|\int_{a^{1-u}b^u}^{a^u b^{1-u}}\left(\ln\frac{b}{s}\right)^{\alpha-1}g(s)\frac{ds}{s}\right|\left|f'(a^{1-u}b^u)\right|a^{1-u}b^u\ln\left(\frac{b}{a}\right)du$$
$$\leq \frac{\ln\left(\frac{b}{a}\right)}{\Gamma(\alpha)}\left[\int_0^1\left|\int_{a^{1-u}b^u}^{a^u b^{1-u}}\left(\ln\frac{b}{s}\right)^{\alpha-1}g(s)\frac{ds}{s}\right|du\right]^{1-\frac{1}{q}}$$
$$\times\left[\int_0^1\left|\int_{a^{1-u}b^u}^{a^u b^{1-u}}\left(\ln\frac{b}{s}\right)^{\alpha-1}g(s)\frac{ds}{s}\right|\left|f'(a^{1-u}b^u)\right|^q\left(a^{1-u}b^u\right)^q du\right]^{\frac{1}{q}}$$
$$\leq \frac{\|g\|_\infty^{1-\frac{1}{q}}\ln^{\alpha\left(1-\frac{1}{q}\right)+1}\left(\frac{b}{a}\right)}{\alpha^{1-\frac{1}{q}}\Gamma(\alpha)}\left[\int_0^{\frac{1}{2}}\left[(1-u)^\alpha-u^\alpha\right]du+\int_{\frac{1}{2}}^1\left[u^\alpha-(1-u)^\alpha\right]du\right]^{1-\frac{1}{q}}$$
$$\times\left[\int_0^1\left|\int_{a^{1-u}b^u}^{a^u b^{1-u}}\left(\ln\frac{b}{s}\right)^{\alpha-1}g(s)\frac{ds}{s}\right|\left[(1-u)\left|f'(a)\right|^q+u\left|f'(b)\right|^q\right]\left(a^{1-u}b^u\right)^q du\right]^{\frac{1}{q}}$$
$$=\frac{\|g\|_\infty^{1-\frac{1}{q}}\ln^{\alpha\left(1-\frac{1}{q}\right)+1}\left(\frac{b}{a}\right)}{\alpha^{1-\frac{1}{q}}\Gamma(\alpha)}\left[\left(1-\frac{1}{2^\alpha}\right)\left(\frac{2}{\alpha+1}\right)\right]^{1-\frac{1}{q}}$$
$$\times\left[\left(\int_0^1\left|\int_{a^{1-u}b^u}^{a^u b^{1-u}}\left(\ln\frac{b}{s}\right)^{\alpha-1}g(s)\frac{ds}{s}\right|(1-u)\left(a^{1-u}b^u\right)^q du\right)\left|f'(a)\right|^q\right.$$
$$\left.+\left(\int_0^1\left|\int_{a^{1-u}b^u}^{a^u b^{1-u}}\left(\ln\frac{b}{s}\right)^{\alpha-1}g(s)\frac{ds}{s}\right|u\left(a^{1-u}b^u\right)^q du\right)\left|f'(b)\right|^q\right]^{\frac{1}{q}}$$
$$\leq \frac{\|g\|_\infty \ln^{\alpha+1}\left(\frac{b}{a}\right)}{\Gamma(\alpha+1)}\left[\left(1-\frac{1}{2^\alpha}\right)\left(\frac{2}{\alpha+1}\right)\right]^{1-\frac{1}{q}}$$

$$\times\left[\left(\begin{array}{c}\int_0^{\frac{1}{2}}\left[(1-u)^{\alpha}-u^{\alpha}\right](1-u)\left(a^{1-u}b^{u}\right)^{q}du+\\ \int_{\frac{1}{2}}^{1}\left[u^{\alpha}-(1-u)^{\alpha}\right](1-u)\left(a^{1-u}b^{u}\right)^{q}du\end{array}\right)\left|f'(a)\right|^{q}\right.$$

$$\left.+\left(\begin{array}{c}\int_0^{\frac{1}{2}}\left[(1-u)^{\alpha}-u^{\alpha}\right]u\left(a^{1-u}b^{u}\right)^{q}du+\\ \int_{\frac{1}{2}}^{1}\left[u^{\alpha}-(1-u)^{\alpha}\right]u\left(a^{1-u}b^{u}\right)^{q}du\end{array}\right)\left|f'(b)\right|^{q}\right]^{\frac{1}{q}}$$

$$=\frac{\|g\|_{\infty}\ln^{\alpha+1}\left(\frac{b}{a}\right)}{\Gamma(\alpha+1)}\left[\left(1-\frac{1}{2^{\alpha}}\right)\left(\frac{2}{\alpha+1}\right)\right]^{1-\frac{1}{q}}$$

$$\times\left[\left(\int_0^{\frac{1}{2}}\left[(1-u)^{\alpha}-u^{\alpha}\right]\left[(1-u)\left(a^{1-u}b^{u}\right)^{q}+u\left(a^{u}b^{1-u}\right)^{q}\right]du\right)\left|f'(a)\right|^{q}\right.$$

$$\left.+\left(\int_0^{\frac{1}{2}}\left[(1-u)^{\alpha}-u^{\alpha}\right]\left[u\left(a^{1-u}b^{u}\right)^{q}+(1-u)\left(a^{u}b^{1-u}\right)^{q}\right]du\right)\left|f'(b)\right|^{q}\right]^{\frac{1}{q}}.$$

This completes the proof. □

Corollary 2.2. *In Theorem 2.3; if one takes $\alpha=1$, $g(x)=1$ and $\alpha=1=g(x)$ respectively, one has the following inequalities:*

$$\left|\frac{f(a)+f(b)}{2}\int_a^b\frac{g(x)}{x}dx-\int_a^b\frac{f(x)g(x)}{x}dx\right| \tag{2.19}$$

$$\leq\|g\|_{\infty}\ln^{2}\left(\frac{b}{a}\right)\left(\frac{1}{2}\right)^{2-\frac{1}{q}}\left[C_3(1)\left|f'(a)\right|^{q}+C_4(1)\left|f'(b)\right|^{q}\right]^{\frac{1}{q}},$$

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2\left(\ln\frac{b}{a}\right)^{\alpha}}\left[J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)\right]\right| \tag{2.20}$$

$$\leq\frac{\ln\left(\frac{b}{a}\right)}{2}\left[\left(1-\frac{1}{2^{\alpha}}\right)\left(\frac{2}{\alpha+1}\right)\right]^{1-\frac{1}{q}}\left[C_3(\alpha)\left|f'(a)\right|^{q}+C_4(\alpha)\left|f'(b)\right|^{q}\right]^{\frac{1}{q}},$$

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{\ln\frac{b}{a}}\int_a^b\frac{f(x)}{x}dx\right|\leq\ln\left(\frac{b}{a}\right)\left(\frac{1}{2}\right)^{2-\frac{1}{q}}\left[C_3(1)\left|f'(a)\right|^{q}+C_4(1)\left|f'(b)\right|^{q}\right]^{\frac{1}{q}}. \tag{2.21}$$

Theorem 2.4. *Let $f:I\subset(0,\infty)\rightarrow\mathbb{R}$ be a differentiable mapping on I° such that $f\in L[a,b]$ where $a,b\in I$ with $a<b$. If $|f'|^{q}, q>1$, is GA-convex on $[a,b]$ and $g:[a,b]\rightarrow\mathbb{R}$ is continuous and geometrically symmetric with respect to $\sqrt{ab}$, then the following inequalities via fractional integrals holds:*

(i) If $\alpha>0$ and $1/p+1/q=1$

$$\left|\left(\frac{f(a)+f(b)}{2}\right)\left[J_{a+}^{\alpha}g(b)+J_{b-}^{\alpha}g(a)\right]-\left[J_{a+}^{\alpha}(fg)(b)+J_{b-}^{\alpha}(fg)(a)\right]\right| \tag{2.22}$$

$$\leq\frac{\|g\|_{\infty}\ln^{\alpha+1-\frac{2}{q}}\left(\frac{b}{a}\right)}{q^{\frac{2}{q}}\Gamma(\alpha+1)}\left[\frac{2}{\alpha p+1}\left(1-\frac{1}{2^{\alpha p}}\right)\right]^{\frac{1}{p}}\left[\begin{array}{c}\left(b^{q}-qa^{q}\ln\frac{b}{a}-a^{q}\right)\left|f'(a)\right|^{q}\\+\left(a^{q}+qb^{q}\ln\frac{b}{a}-b^{q}\right)\left|f'(b)\right|^{q}\end{array}\right]^{\frac{1}{q}},$$

(ii) If $0<\alpha\leq 1$ *and* $1/p+1/q=1$

$$\left|\left(\frac{f(a)+f(b)}{2}\right)\left[J_{a+}^{\alpha}g(b)+J_{b-}^{\alpha}g(a)\right]-\left[J_{a+}^{\alpha}\left(fg\right)(b)+J_{b-}^{\alpha}\left(fg\right)(a)\right]\right| \tag{2.23}$$
$$\leq \frac{\|g\|_{\infty}\ln^{\alpha+1-\frac{2}{q}}\left(\frac{b}{a}\right)}{q^{\frac{2}{q}}\Gamma(\alpha+1)}\left[\frac{1}{\alpha p+1}\right]^{\frac{1}{p}}$$
$$\times\left[\left(b^{q}-qa^{q}\ln\frac{b}{a}-a^{q}\right)\left|f'(a)\right|^{q}+\left(a^{q}+qb^{q}\ln\frac{b}{a}-b^{q}\right)\left|f'(b)\right|^{q}\right]^{\frac{1}{q}}.$$

Proof. (i) Using Lemma 2.2, (2.9), (2.10), Hölder's inequality and GA-convexity of $|f'|^{q}$, we have

$$\left|\left(\frac{f(a)+f(b)}{2}\right)\left[J_{a+}^{\alpha}g(b)+J_{b-}^{\alpha}g(a)\right]-\left[J_{a+}^{\alpha}\left(fg\right)(b)+J_{b-}^{\alpha}\left(fg\right)(a)\right]\right|$$
$$\leq\frac{1}{\Gamma(\alpha)}\int_{0}^{1}\left|\int_{a^{1-u}b^{u}}^{a^{u}b^{1-u}}\left(\ln\frac{b}{s}\right)^{\alpha-1}g(s)\frac{ds}{s}\right|\left|f'(a^{1-u}b^{u})\right|a^{1-u}b^{u}\ln\left(\frac{b}{a}\right)du$$
$$\leq\frac{\ln\left(\frac{b}{a}\right)}{\Gamma(\alpha)}\left[\int_{0}^{1}\left|\int_{a^{1-u}b^{u}}^{a^{u}b^{1-u}}\left(\ln\frac{b}{s}\right)^{\alpha-1}g(s)\frac{ds}{s}\right|^{p}du\right]^{\frac{1}{p}}\left[\int_{0}^{1}\left|f'(a^{1-u}b^{u})\right|^{q}\left(a^{1-u}b^{u}\right)^{q}du\right]^{\frac{1}{q}}$$
$$\leq\frac{\|g\|_{\infty}\ln^{\alpha+1}\left(\frac{b}{a}\right)}{\Gamma(\alpha+1)}\left[\int_{0}^{\frac{1}{2}}\left[(1-u)^{\alpha}-u^{\alpha}\right]^{p}du+\int_{\frac{1}{2}}^{1}\left[u^{\alpha}-(1-u)^{\alpha}\right]^{p}du\right]^{\frac{1}{p}}$$
$$\times\left[\int_{0}^{1}\left[(1-u)\left|f'(a)\right|^{q}+u\left|f'(b)\right|^{q}\right]\left(a^{1-u}b^{u}\right)^{q}du\right]^{\frac{1}{q}}$$
$$=\frac{\|g\|_{\infty}\ln^{\alpha+1-\frac{2}{q}}\left(\frac{b}{a}\right)}{q^{\frac{2}{q}}\Gamma(\alpha+1)}\left[\int_{0}^{\frac{1}{2}}\left[(1-u)^{\alpha}-u^{\alpha}\right]^{p}du+\int_{\frac{1}{2}}^{1}\left[u^{\alpha}-(1-u)^{\alpha}\right]^{p}du\right]^{\frac{1}{p}}$$
$$\times\left[\left(b^{q}-qa^{q}\ln\frac{b}{a}-a^{q}\right)\left|f'(a)\right|^{q}+\left(a^{q}+qb^{q}\ln\frac{b}{a}-b^{q}\right)\left|f'(b)\right|^{q}\right]^{\frac{1}{q}} \tag{2.24}$$
$$\leq\frac{\|g\|_{\infty}\ln^{\alpha+1-\frac{2}{q}}\left(\frac{b}{a}\right)}{q^{\frac{2}{q}}\Gamma(\alpha+1)}\left[\int_{0}^{\frac{1}{2}}(1-u)^{\alpha p}-u^{\alpha p}du+\int_{\frac{1}{2}}^{1}u^{\alpha p}-(1-u)^{\alpha p}du\right]^{\frac{1}{p}}$$
$$\times\left[\left(b^{q}-qa^{q}\ln\frac{b}{a}-a^{q}\right)\left|f'(a)\right|^{q}+\left(a^{q}+qb^{q}\ln\frac{b}{a}-b^{q}\right)\left|f'(b)\right|^{q}\right]^{\frac{1}{q}}$$
$$=\frac{\|g\|_{\infty}\ln^{\alpha+1-\frac{2}{q}}\left(\frac{b}{a}\right)}{q^{\frac{2}{q}}\Gamma(\alpha+1)}\left[\frac{2}{\alpha p+1}\left(1-\frac{1}{2^{\alpha p}}\right)\right]^{\frac{1}{p}}$$
$$\times\left[\left(b^{q}-qa^{q}\ln\frac{b}{a}-a^{q}\right)\left|f'(a)\right|^{q}+\left(a^{q}+qb^{q}\ln\frac{b}{a}-b^{q}\right)\left|f'(b)\right|^{q}\right]^{\frac{1}{q}}.$$

In here we use

$$\left[(1-t)^{\alpha}-t^{\alpha}\right]^{p}\leq(1-t)^{\alpha p}-t^{\alpha p},$$

for $t\in[0,1/2]$ and

$$\left[t^{\alpha}-(1-t)^{\alpha}\right]^{p}\leq t^{\alpha p}-(1-t)^{\alpha p},$$

for $t\in[1/2,1]$, which follows from

$$(A-B)^{q}\leq A^{q}-B^{q},$$

for any $A\geq B\geq 0$ and $q\geq 1$. Hence the inequality (2.22) is proved.

(ii) The inequality (2.23) could be prove by using (2.24) and Lemma 1.1. □

Corollary 2.3. *In Theorem 2.4;*

(i) In (2.16); if one takes $\alpha=1$, $g(x)=1$ and $\alpha=1=g(x)$ respectively, one has the following inequalities:

$$\left|\frac{f(a)+f(b)}{2}\int_a^b\frac{g(x)}{x}dx-\int_a^b\frac{f(x)g(x)}{x}dx\right| \tag{2.25}$$
$$\leq\frac{\|g\|_\infty\ln^{2-\frac{2}{q}}\left(\frac{b}{a}\right)}{2q^{\frac{2}{q}}}\left[\frac{2}{p+1}\left(1-\frac{1}{2^p}\right)\right]^{\frac{1}{p}}\left[\begin{array}{c}\left(b^q-qa^q\ln\frac{b}{a}-a^q\right)|f'(a)|^q\\+\left(a^q+qb^q\ln\frac{b}{a}-b^q\right)|f'(b)|^q\end{array}\right]^{\frac{1}{q}},$$

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2\left(\ln\frac{b}{a}\right)^\alpha}\left[J_{a+}^\alpha f(b)+J_{b-}^\alpha f(a)\right]\right| \tag{2.26}$$
$$\leq\frac{\ln^{1-\frac{2}{q}}\left(\frac{b}{a}\right)}{2q^{\frac{2}{q}}}\left[\frac{2}{\alpha p+1}\left(1-\frac{1}{2^{\alpha p}}\right)\right]^{\frac{1}{p}}\left[\begin{array}{c}\left(b^q-qa^q\ln\frac{b}{a}-a^q\right)|f'(a)|^q\\+\left(a^q+qb^q\ln\frac{b}{a}-b^q\right)|f'(b)|^q\end{array}\right]^{\frac{1}{q}},$$

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{\ln\frac{b}{a}}\int_a^b\frac{f(x)}{x}dx\right| \tag{2.27}$$
$$\leq\frac{\ln^{1-\frac{2}{q}}\left(\frac{b}{a}\right)}{2q^{\frac{2}{q}}}\left[\frac{2}{p+1}\left(1-\frac{1}{2^p}\right)\right]^{\frac{1}{p}}\left[\begin{array}{c}\left(b^q-qa^q\ln\frac{b}{a}-a^q\right)|f'(a)|^q\\+\left(a^q+qb^q\ln\frac{b}{a}-b^q\right)|f'(b)|^q\end{array}\right]^{\frac{1}{q}}.$$

(ii) In (2.17); if one takes $\alpha=1$, $g(x)=1$ and $\alpha=1=g(x)$ respectively, one has the following inequalities:

$$\left|\frac{f(a)+f(b)}{2}\int_a^b\frac{g(x)}{x}dx-\int_a^b\frac{f(x)g(x)}{x}dx\right| \tag{2.28}$$
$$\leq\frac{\|g\|_\infty\ln^{2-\frac{2}{q}}\left(\frac{b}{a}\right)}{2q^{\frac{2}{q}}}\left[\frac{1}{p+1}\right]^{\frac{1}{p}}$$
$$\times\left[\left(b^q-qa^q\ln\frac{b}{a}-a^q\right)|f'(a)|^q+\left(a^q+qb^q\ln\frac{b}{a}-b^q\right)|f'(b)|^q\right]^{\frac{1}{q}},$$

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2\left(\ln\frac{b}{a}\right)^\alpha}\left[J_{a+}^\alpha f(b)+J_{b-}^\alpha f(a)\right]\right| \tag{2.29}$$
$$\leq\frac{\ln^{1-\frac{2}{q}}\left(\frac{b}{a}\right)}{2q^{\frac{2}{q}}}\left[\frac{1}{\alpha p+1}\right]^{\frac{1}{p}}\left[\left(b^q-qa^q\ln\frac{b}{a}-a^q\right)|f'(a)|^q+\left(a^q+qb^q\ln\frac{b}{a}-b^q\right)|f'(b)|^q\right]^{\frac{1}{q}},$$

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{\ln\frac{b}{a}}\int_a^b\frac{f(x)}{x}dx\right| \tag{2.30}$$

$$\leq \frac{\ln^{1-\frac{2}{q}}\left(\frac{b}{a}\right)}{2q^{\frac{2}{q}}}\left[\frac{1}{p+1}\right]^{\frac{1}{p}}\left[\left(b^q-qa^q\ln\frac{b}{a}-a^q\right)\left|f'(a)\right|^q+\left(a^q+qb^q\ln\frac{b}{a}-b^q\right)\left|f'(b)\right|^q\right]^{\frac{1}{q}}.$$

Theorem 2.5. *Let $f:[a,b]\subseteq\mathbb{R}_+\to\mathbb{R}$ be a GA-convex function, $\alpha>0$ and $f\in L[a,b]$, then the following inequalities for fractional integrals hold:*

$$f\left(\sqrt{ab}\right)\leq\frac{\Gamma(\alpha+1)}{2^{1-\alpha}\left(\ln\frac{b}{a}\right)^{\alpha}}\left[J^{\alpha}_{\sqrt{ab}-}f(a)+J^{\alpha}_{\sqrt{ab}+}f(b)\right]\leq\frac{f(a)+f(b)}{2}. \tag{2.31}$$

Proof. Using GA-convexity of the function f on $[a,b]$, we have

$$f\left(\sqrt{xy}\right)\leq\frac{f(x)+f(y)}{2}.$$

for all $x,y\in[a,b]$. Choosing $x=a^{1-t}b^t$, $y=a^tb^{1-t}$, we have

$$f\left(\sqrt{ab}\right)\leq\frac{f\left(a^{1-t}b^t\right)+f\left(a^tb^{1-t}\right)}{2}. \tag{2.32}$$

Multiplying both sides of (2.32) by $t^{\alpha-1}$, then integrating the resulting inequality with respect to t over $\left[0,\frac{1}{2}\right]$, we have

$$f\left(\sqrt{ab}\right)\leq\frac{\alpha}{2^{1-\alpha}}\left[\int_0^{\frac{1}{2}}t^{\alpha-1}f\left(a^{1-t}b^t\right)dt+\int_0^{\frac{1}{2}}t^{\alpha-1}f\left(a^tb^{1-t}\right)dt\right]$$

$$=\frac{\alpha}{2^{1-\alpha}}\left[\int_a^{\sqrt{ab}}\left(\frac{\ln\frac{x}{a}}{\ln\frac{b}{a}}\right)^{\alpha-1}f(x)\frac{dx}{x\ln\frac{b}{a}}+\int_{\sqrt{ab}}^{b}\left(\frac{\ln\frac{b}{x}}{\ln\frac{b}{a}}\right)^{\alpha-1}f(x)\frac{dx}{x\ln\frac{b}{a}}\right]$$

$$=\frac{\Gamma(\alpha+1)}{2^{1-\alpha}\left(\ln\frac{b}{a}\right)^{\alpha}}\left[\frac{1}{\Gamma(\alpha)}\int_a^{\sqrt{ab}}\left(\ln\frac{x}{a}\right)^{\alpha-1}f(x)\frac{dx}{x}+\frac{1}{\Gamma(\alpha)}\int_{\sqrt{ab}}^{b}\left(\ln\frac{b}{x}\right)^{\alpha-1}f(x)\frac{dx}{x}\right]$$

$$=\frac{\Gamma(\alpha+1)}{2^{1-\alpha}\left(\ln\frac{b}{a}\right)^{\alpha}}\left[J^{\alpha}_{\sqrt{ab}-}f(a)+J^{\alpha}_{\sqrt{ab}+}f(b)\right].$$

the first inequality in (2.31) is proved.

For the proof of the second inequality in (2.31), we first note that if f is a convex function, then for $t\in[0,1]$, it yields

$$f\left(a^{1-t}b^t\right)\leq(1-t)f(a)+tf(b),$$

and

$$f\left(a^tb^{1-t}\right)\leq tf(a)+(1-t)f(b).$$

By adding these inequalities, we have

$$f\left(a^{1-t}b^t\right)+f\left(a^tb^{1-t}\right)\leq f(a)+f(b). \tag{2.33}$$

Multiplying both sides of (2.33) by $t^{\alpha-1}$ and integrating the resulting inequality with respect to t over $\left[0,\frac{1}{2}\right]$, we have

$$\int_0^{\frac{1}{2}}t^{\alpha-1}f\left(a^{1-t}b^t\right)dt+\int_0^{\frac{1}{2}}t^{\alpha-1}f\left(a^tb^{1-t}\right)dt\leq[f(a)+f(b)]\int_0^{\frac{1}{2}}t^{\alpha-1}dt.$$

Using Lemma 2.1, we have

$$\frac{\Gamma(\alpha+1)}{2^{1-\alpha}\left(\ln\frac{b}{a}\right)^{\alpha}}\left[J^{\alpha}_{\sqrt{ab}-}f(a)+J^{\alpha}_{\sqrt{ab}+}f(b)\right]\leq\frac{f(a)+f(b)}{2},$$

which is the proof of the second inequality in (2.31).The proof is completed. □

Theorem 2.6. *Let $f:[a,b]\subseteq\mathbb{R}_{+}\to\mathbb{R}$ be a GA-convex function, $\alpha>0$ and $f\in L[a,b]$. If $g:[a,b]\to\mathbb{R}$ is nonnegative,integrable and geometrically symmetric with respect to $\sqrt{ab}$, then the following inequalities for fractional integrals hold:*

$$f\left(\sqrt{ab}\right)\left[J^{\alpha}_{\sqrt{ab}-}g(a)+J^{\alpha}_{\sqrt{ab}+}g(b)\right]\leq\left[J^{\alpha}_{\sqrt{ab}-}(fg)(a)+J^{\alpha}_{\sqrt{ab}+}(fg)(b)\right] \tag{2.34}$$

$$\leq\frac{f(a)+f(b)}{2}\left[J^{\alpha}_{\sqrt{ab}-}g(a)+J^{\alpha}_{\sqrt{ab}+}g(b)\right].$$

Proof. Using GA-convexity of the function f on $[a,b]$, we have

$$f\left(\sqrt{ab}\right)=f\left(\sqrt{a^{1-t}b^{t}a^{t}b^{1-t}}\right)\leq\frac{f\left(a^{1-t}b^{t}\right)+f\left(a^{t}b^{1-t}\right)}{2}, \tag{2.35}$$

for all $t\in[0,1]$. Multiplying both sides of (2.35) by $2t^{\alpha-1}g\left(a^{1-t}b^{t}\right)$ then integrating the resulting inequality with respect to t over $\left[0,\frac{1}{2}\right]$, we have

$$2f\left(\sqrt{ab}\right)\int_0^{\frac{1}{2}}t^{\alpha-1}g\left(a^{1-t}b^{t}\right)dt$$

$$\leq\int_0^{\frac{1}{2}}t^{\alpha-1}\left[f\left(a^{1-t}b^{t}\right)+f\left(a^{t}b^{1-t}\right)\right]g\left(a^{1-t}b^{t}\right)dt$$

$$=\int_0^{\frac{1}{2}}t^{\alpha-1}f\left(a^{1-t}b^{t}\right)g\left(a^{1-t}b^{t}\right)dt+\int_0^{\frac{1}{2}}t^{\alpha-1}f\left(a^{t}b^{1-t}\right)g\left(a^{1-t}b^{t}\right)dt.$$

Setting $x=a^{1-t}b^{t}$, and $dx=a^{1-t}b^{t}\ln\left(\frac{b}{a}\right)dt$ gives

$$\frac{2}{\left(\ln\frac{b}{a}\right)^{\alpha}}f\left(\sqrt{ab}\right)\int_a^{\sqrt{ab}}\left(\ln\frac{x}{a}\right)^{\alpha-1}g(x)\frac{dx}{x}$$

$$\leq\frac{1}{\left(\ln\frac{b}{a}\right)^{\alpha}}\left\{\int_a^{\sqrt{ab}}\left(\ln\frac{x}{a}\right)^{\alpha-1}f(x)g(x)\frac{dx}{x}+\int_a^{\sqrt{ab}}\left(\ln\frac{x}{a}\right)^{\alpha-1}f\left(\frac{ab}{x}\right)g(x)\frac{dx}{x}\right\}$$

$$=\frac{1}{\left(\ln\frac{b}{a}\right)^{\alpha}}\left\{\int_a^{\sqrt{ab}}\left(\ln\frac{x}{a}\right)^{\alpha-1}f(x)g(x)\frac{dx}{x}+\int_{\sqrt{ab}}^{b}\left(\ln\frac{b}{u}\right)^{\alpha-1}f(u)g\left(\frac{ab}{u}\right)\frac{du}{u}\right\}$$

$$=\frac{1}{\left(\ln\frac{b}{a}\right)^{\alpha}}\left\{\int_a^{\sqrt{ab}}\left(\ln\frac{x}{a}\right)^{\alpha-1}f(x)g(x)\frac{dx}{x}+\int_{\sqrt{ab}}^{b}\left(\ln\frac{b}{u}\right)^{\alpha-1}f(u)g(u)\frac{du}{u}\right\}.$$

Therefore, using Lemma 2.1, we have

$$\frac{\Gamma(\alpha)}{\left(\ln\frac{b}{a}\right)^{\alpha}}f\left(\sqrt{ab}\right)\left[J^{\alpha}_{\sqrt{ab}-}g(a)+J^{\alpha}_{\sqrt{ab}+}g(b)\right]\leq\frac{\Gamma(\alpha)}{\left(\ln\frac{b}{a}\right)^{\alpha}}\left[J^{\alpha}_{\sqrt{ab}-}(fg)(a)+J^{\alpha}_{\sqrt{ab}+}(fg)(b)\right],$$

and the first inequality is proved.

For the proof of the second inequality in (2.34) we first note that if f is a GA-convex function, then, for all $t\in[0,1]$, it yields

$$f\left(a^{1-t}b^{t}\right)+f\left(a^{t}b^{1-t}\right)\leq f\left(a\right)+f\left(b\right). \tag{2.36}$$

Then multiplying both sides of (2.36) by $t^{\alpha-1}g\left(a^{1-t}b^{t}\right)$ and integrating the resulting inequality with respect to t over $\left[0,\frac{1}{2}\right]$, we have

$$\int_0^{\frac{1}{2}}t^{\alpha-1}f\left(a^{1-t}b^{t}\right)g\left(a^{1-t}b^{t}\right)dt+\int_0^{\frac{1}{2}}t^{\alpha-1}f\left(a^{t}b^{1-t}\right)g\left(a^{1-t}b^{t}\right)dt$$
$$\leq\left[f(a)+f(b)\right]\int_0^{\frac{1}{2}}t^{\alpha-1}g\left(a^{1-t}b^{t}\right)dt.$$

Using Lemma 2.1, we have

$$\frac{\Gamma(\alpha)}{\left(\ln\frac{b}{a}\right)^{\alpha}}\left[J^{\alpha}_{\sqrt{ab}-}\left(fg\right)\left(a\right)+J^{\alpha}_{\sqrt{ab}+}\left(fg\right)\left(b\right)\right]$$
$$\leq\frac{\Gamma(\alpha)}{\left(\ln\frac{b}{a}\right)^{\alpha}}\left(\frac{f\left(a\right)+f\left(b\right)}{2}\right)\left[J^{\alpha}_{\sqrt{ab}-}g\left(a\right)+J^{\alpha}_{\sqrt{ab}+}g\left(b\right)\right],$$

which is the proof of the second inequality in (2.34).The proof is completed. □

Remark 2.2. In Theorem 2.6,

(1) if one takes $\alpha=1$, one recaptures the inequality [5, Theorem 6],
(2) if one take $g(x)=1$, one recaptures (2.31) of Theorem 2.5.

Lemma 2.3. *Let $f:[a,b]\subseteq\mathbb{R}_{+}\to\mathbb{R}$ be a differentiable mapping on (a,b) with $a<b$ and $f'\in L\left[a,b\right]$. If $g:[a,b]\to\mathbb{R}$ is integrable and geometrically symmetric with respect to $\sqrt{ab}$ then the following equality for fractional integrals holds:*

$$f\left(\sqrt{ab}\right)\left[J^{\alpha}_{\sqrt{ab}-}g\left(a\right)+J^{\alpha}_{\sqrt{ab}+}g\left(b\right)\right]-\left[J^{\alpha}_{\sqrt{ab}-}\left(fg\right)\left(a\right)+J^{\alpha}_{\sqrt{ab}+}\left(fg\right)\left(b\right)\right] \tag{2.37}$$
$$=\frac{1}{\Gamma(\alpha)}\left[\int_a^{\sqrt{ab}}\left(\int_a^t\left(\ln\frac{s}{a}\right)^{\alpha-1}g(s)\frac{ds}{s}\right)f'\left(t\right)dt+\int_{\sqrt{ab}}^b\left(\int_t^b\left(\ln\frac{b}{s}\right)^{\alpha-1}g(s)\frac{ds}{s}\right)f'\left(t\right)dt\right]$$

with $\alpha>0$.

Proof. It suffices to note that

$$I=\int_a^{\sqrt{ab}}\left(\int_a^t\left(\ln\frac{s}{a}\right)^{\alpha-1}g(s)\frac{ds}{s}\right)f'\left(t\right)dt+\int_{\sqrt{ab}}^b\left(\int_t^b\left(\ln\frac{b}{s}\right)^{\alpha-1}g(s)\frac{ds}{s}\right)f'\left(t\right)dt=I_1+I_2. \tag{2.38}$$

By integration by parts we get

$$I_1=\left(\int_a^t\left(\ln\frac{s}{a}\right)^{\alpha-1}g(s)\frac{ds}{s}\right)f(t)\Bigg|_a^{\sqrt{ab}}-\int_a^{\sqrt{ab}}\left(\ln\frac{t}{a}\right)^{\alpha-1}g(t)f(t)\frac{dt}{t} \tag{2.39}$$
$$=\left(\int_a^{\sqrt{ab}}\left(\ln\frac{b}{s}\right)^{\alpha-1}g(s)\frac{ds}{s}\right)f(\sqrt{ab})-\int_a^{\sqrt{ab}}\left(\ln\frac{b}{t}\right)^{\alpha-1}(fg)(t)\frac{dt}{t}$$
$$=\Gamma(\alpha)\left[f(\sqrt{ab})J^{\alpha}_{\sqrt{ab}-}g(b)-J^{\alpha}_{\sqrt{ab}-}(fg)(b)\right],$$

and similarly

$$I_2 = \left(\int_t^b \left(\ln \frac{b}{s} \right)^{\alpha-1} g(s) \frac{ds}{s} \right) f(t) \Big|_{\sqrt{ab}}^{b} - \int_{\sqrt{ab}}^{b} \left(\ln \frac{b}{t} \right)^{\alpha-1} g(t) f(t) \frac{dt}{t} \tag{2.40}$$

$$= \left(\int_{\sqrt{ab}}^{b} \left(\ln \frac{s}{a} \right)^{\alpha-1} g(s) \frac{ds}{s} \right) f(\sqrt{ab}) - \int_a^b \left(\ln \frac{t}{a} \right)^{\alpha-1} (fg)(t) \frac{dt}{t}$$

$$= \Gamma(\alpha) \left[f\left(\sqrt{ab}\right) J^{\alpha}_{\sqrt{ab}+} g(b) - J^{\alpha}_{\sqrt{ab}+} (fg)(b) \right].$$

Using (2.39) and (2.40) in (2.38), than multiplying the both sides of the resulting inequality by $(\Gamma(\alpha))^{-1}$, we have (2.37). This completes the proof. □

Theorem 2.7. *Let $f : I \subseteq \mathbb{R}_+ \to \mathbb{R}$ be a differentiable mapping on I° and $f' \in L[a,b]$ where $a, b \in I$ with $a < b$ and $\alpha > 0$. If $|f'|$ is GA-convex on $[a,b]$ and $g : [a,b] \to \mathbb{R}$ is continuous and geometrically symmetric with respect to $\sqrt{ab}$, then the following inequality for fractional integrals holds*

$$\left| f\left(\sqrt{ab}\right) \left[J^{\alpha}_{\sqrt{ab}-} g(a) + J^{\alpha}_{\sqrt{ab}+} g(b) \right] - \left[J^{\alpha}_{\sqrt{ab}-} (fg)(a) + J^{\alpha}_{\sqrt{ab}+} (fg)(b) \right] \right| \tag{2.41}$$

$$\leq \frac{\|g\|_\infty \ln^{\alpha+1}\left(\frac{b}{a}\right)}{\Gamma(\alpha+1)} \left[T_1(\alpha) \left| f'(a) \right| + T_2(\alpha) \left| f'(b) \right| \right],$$

where

$$T_1(\alpha) = \int_0^{\frac{1}{2}} u^\alpha \left[(1-u) \left(a^{1-u} b^u \right) + u \left(a^u b^{1-u} \right) \right] du,$$

$$T_2(\alpha) = \int_0^{\frac{1}{2}} u^\alpha \left[u \left(a^{1-u} b^u \right) + (1-u) \left(a^u b^{1-u} \right) \right] du,$$

Proof. Using Lemma 2.2, we have

$$\left| f\left(\sqrt{ab}\right) \left[J^{\alpha}_{\sqrt{ab}-} g(a) + J^{\alpha}_{\sqrt{ab}+} g(b) \right] - \left[J^{\alpha}_{\sqrt{ab}-} (fg)(a) + J^{\alpha}_{\sqrt{ab}+} (fg)(b) \right] \right|$$

$$\leq \frac{1}{\Gamma(\alpha)} \left[\begin{array}{c} \int_a^{\sqrt{ab}} \left(\int_a^t \left(\ln \frac{s}{a} \right)^{\alpha-1} |g(s)| \frac{ds}{s} \right) |f'(t)| \, dt \\ + \int_{\sqrt{ab}}^b \left(\int_t^b \left(\ln \frac{b}{s} \right)^{\alpha-1} |g(s)| \frac{ds}{s} \right) |f'(t)| \, dt \end{array} \right]$$

$$\leq \frac{\|g\|_\infty}{\Gamma(\alpha)} \left[\begin{array}{c} \int_a^{\sqrt{ab}} \left(\int_a^t \left(\ln \frac{s}{a} \right)^{\alpha-1} \frac{ds}{s} \right) |f'(t)| \, dt \\ + \int_{\sqrt{ab}}^b \left(\int_t^b \left(\ln \frac{b}{s} \right)^{\alpha-1} \frac{ds}{s} \right) |f'(t)| \, dt \end{array} \right].$$

Setting $t = a^{1-u} b^u$ and $dt = a^{1-u} b^u \ln\left(\frac{b}{a}\right) du$ gives

$$\left| f\left(\sqrt{ab}\right) \left[J^{\alpha}_{\sqrt{ab}-} g(a) + J^{\alpha}_{\sqrt{ab}+} g(b) \right] - \left[J^{\alpha}_{\sqrt{ab}-} (fg)(a) + J^{\alpha}_{\sqrt{ab}+} (fg)(b) \right] \right| \tag{2.42}$$

$$\leq \frac{\|g\|_\infty}{\Gamma(\alpha)} \left[\begin{array}{c} \int_0^{\frac{1}{2}} \left(\int_a^{a^{1-u}b^u} \left(\ln \frac{s}{a} \right)^{\alpha-1} \frac{ds}{s} \right) \left| f'\left(a^{1-u}b^u\right) \right| \left(a^{1-u}b^u\right) \ln\left(\frac{b}{a}\right) du \\ + \int_{\frac{1}{2}}^1 \left(\int_{a^{1-u}b^u}^b \left(\ln \frac{b}{s} \right)^{\alpha-1} \frac{ds}{s} \right) \left| f'\left(a^{1-u}b^u\right) \right| \left(a^{1-u}b^u\right) \ln\left(\frac{b}{a}\right) du \end{array} \right]$$

$$= \frac{\|g\|_{\infty}}{\Gamma(\alpha)} \begin{bmatrix} \int_0^{\frac{1}{2}} \left(\frac{\left(\ln \frac{s}{a}\right)^{\alpha}}{\alpha} \Big|_a^{a^{1-u}b^u} \right) \left|f'\left(a^{1-u}b^u\right)\right| \left(a^{1-u}b^u\right) \ln\left(\frac{b}{a}\right) du \\ + \int_{\frac{1}{2}}^1 \left(\frac{-\left(\ln \frac{b}{s}\right)^{\alpha}}{\alpha} \Big|_{a^{1-u}b^u}^{b} \right) \left|f'\left(a^{1-u}b^u\right)\right| \left(a^{1-u}b^u\right) \ln\left(\frac{b}{a}\right) du \end{bmatrix}$$

$$= \frac{\|g\|_{\infty} \left(\ln \frac{b}{a}\right)^{\alpha+1}}{\Gamma(\alpha+1)} \begin{bmatrix} \int_0^{\frac{1}{2}} u^{\alpha} \left|f'\left(a^{1-u}b^u\right)\right| \left(a^{1-u}b^u\right) du \\ + \int_{\frac{1}{2}}^1 (1-u)^{\alpha} \left|f'\left(a^{1-u}b^u\right)\right| \left(a^{1-u}b^u\right) du \end{bmatrix}$$

Using GA-convexity of $|f'|$ on $[a,b]$, we have

$$\left|f'(a^{1-u}b^u)\right| \leq (1-u)\left|f'(a)\right| + u\left|f'(b)\right|, \tag{2.43}$$

for all $u \in [0,1]$. If we use (2.43) in (2.42), we have

$$\left| f\left(\sqrt{ab}\right)\left[J^{\alpha}_{\sqrt{ab}-} g(a) + J^{\alpha}_{\sqrt{ab}+} g(b)\right] - \left[J^{\alpha}_{\sqrt{ab}-}(fg)(a) + J^{\alpha}_{\sqrt{ab}+}(fg)(b)\right]\right|$$

$$\leq \frac{\|g\|_{\infty} \left(\ln \frac{b}{a}\right)^{\alpha+1}}{\Gamma(\alpha+1)} \begin{bmatrix} \int_0^{\frac{1}{2}} u^{\alpha} \left[(1-u)\left|f'(a)\right| + u\left|f'(b)\right|\right] \left(a^{1-u}b^u\right) du \\ + \int_{\frac{1}{2}}^1 (1-u)^{\alpha} \left[(1-u)\left|f'(a)\right| + u\left|f'(b)\right|\right] \left(a^{1-u}b^u\right) du \end{bmatrix}$$

$$= \frac{\|g\|_{\infty} \left(\ln \frac{b}{a}\right)^{\alpha+1}}{\Gamma(\alpha+1)} \begin{bmatrix} \int_0^{\frac{1}{2}} u^{\alpha} \left[(1-u)\left|f'(a)\right| + u\left|f'(b)\right|\right] \left(a^{1-u}b^u\right) du \\ + \int_0^{\frac{1}{2}} u^{\alpha} \left[u\left|f'(a)\right| + (1-u)\left|f'(b)\right|\right] \left(a^{u}b^{1-u}\right) du \end{bmatrix}$$

$$= \frac{\|g\|_{\infty} \left(\ln \frac{b}{a}\right)^{\alpha+1}}{\Gamma(\alpha+1)} \begin{bmatrix} \left(\int_0^{\frac{1}{2}} u^{\alpha} \left[(1-u)\left(a^{1-u}b^u\right) + u\left(a^{u}b^{1-u}\right)\right] du\right) \left|f'(a)\right| \\ + \left(\int_0^{\frac{1}{2}} u^{\alpha} \left[u\left(a^{1-u}b^u\right) + (1-u)\left(a^{u}b^{1-u}\right)\right] du\right) \left|f'(b)\right| \end{bmatrix}.$$

This completes the proof. □

Corollary 2.4. *In Theorem 2.7; if one takes $\alpha = 1$, $g(x) = 1$ and $\alpha = 1 = g(x)$ respectively, one has the following inequalities:*

$$\left| f\left(\sqrt{ab}\right) \int_a^b \frac{g(x)}{x} dx - \int_a^b \frac{f(x)g(x)}{x} dx \right| \leq \|g\|_{\infty} \ln^2\left(\frac{b}{a}\right) \left[T_1(1)\left|f'(a)\right| + T_2(1)\left|f'(b)\right|\right], \tag{2.44}$$

$$\left| f\left(\sqrt{ab}\right) - \frac{\Gamma(\alpha+1)}{2^{1-\alpha}\left(\ln \frac{b}{a}\right)^{\alpha}} \left[J^{\alpha}_{\sqrt{ab}-} f(a) + J^{\alpha}_{\sqrt{ab}+} f(b)\right] \right|$$

$$\leq \frac{\ln\left(\frac{b}{a}\right)}{2^{1-\alpha}} \left[T_1(\alpha)\left|f'(a)\right| + T_2(\alpha)\left|f'(b)\right|\right], \tag{2.45}$$

$$\left| f\left(\sqrt{ab}\right) - \frac{1}{\ln \frac{b}{a}} \int_a^b \frac{f(x)}{x} dx \right| \leq \ln\left(\frac{b}{a}\right) \left[T_1(1)\left|f'(a)\right| + T_2(1)\left|f'(b)\right|\right]. \tag{2.46}$$

Theorem 2.8. *Let $f : I \subseteq \mathbb{R}_+ \rightarrow \mathbb{R}$ be a differentiable mapping on I° and $f' \in L[a,b]$ where $a, b \in I$ with $a < b$ and $\alpha > 0$. If $|f'|^q, q \geq 1$, is GA-convex on $[a,b]$ and $g : [a,b] \rightarrow \mathbb{R}$ is* *continuous and geometrically symmetric with respect to $\sqrt{ab}$, then the following inequalities for fractional integrals hold:*

$$\left|f\left(\sqrt{ab}\right)\left[J^{\alpha}_{\sqrt{ab}-}g\left(a\right)+J^{\alpha}_{\sqrt{ab}+}g\left(b\right)\right]-\left[J^{\alpha}_{\sqrt{ab}-}\left(fg\right)\left(a\right)+J^{\alpha}_{\sqrt{ab}+}\left(fg\right)\left(b\right)\right]\right| \tag{2.47}$$
$$\leq \frac{\left\|g\right\|_{\infty}\ln^{\alpha+1}\left(\frac{b}{a}\right)}{2^{(\alpha+1)\left(1-\frac{1}{q}\right)}\left(\alpha+1\right)^{1-\frac{1}{q}}\Gamma(\alpha+1)}\left[\begin{array}{c}\left[T_3(\alpha)\left|f'\left(a\right)\right|^q+T_4(\alpha)\left|f'\left(b\right)\right|^q\right]^{\frac{1}{q}}\\ +\left[T_5(\alpha)\left|f'\left(a\right)\right|^q+T_6(\alpha)\left|f'\left(b\right)\right|^q\right]^{\frac{1}{q}}\end{array}\right],$$

where

$$T_3(\alpha)=\int_0^{\frac{1}{2}}u^{\alpha}\left(1-u\right)\left(a^{1-u}b^{u}\right)^q du,\quad T_4(\alpha)=\int_0^{\frac{1}{2}}u^{\alpha+1}\left(a^{1-u}b^{u}\right)^q du,$$
$$T_5(\alpha)=\int_0^{\frac{1}{2}}u^{\alpha+1}\left(a^{u}b^{1-u}\right)^q du,\quad T_6(\alpha)=\int_0^{\frac{1}{2}}u^{\alpha}\left(1-u\right)\left(a^{u}b^{1-u}\right)^q du.$$

Proof. Using Lemma 2.2, we have

$$\left|f\left(\sqrt{ab}\right)\left[J^{\alpha}_{\sqrt{ab}-}g\left(a\right)+J^{\alpha}_{\sqrt{ab}+}g\left(b\right)\right]-\left[J^{\alpha}_{\sqrt{ab}-}\left(fg\right)\left(a\right)+J^{\alpha}_{\sqrt{ab}+}\left(fg\right)\left(b\right)\right]\right|$$
$$\leq \frac{1}{\Gamma(\alpha)}\left[\begin{array}{c}\int_a^{\sqrt{ab}}\left(\int_a^t\left(\ln\frac{s}{a}\right)^{\alpha-1}\left|g(s)\right|\frac{ds}{s}\right)\left|f'\left(t\right)\right|dt\\ +\int_{\sqrt{ab}}^b\left(\int_t^b\left(\ln\frac{b}{s}\right)^{\alpha-1}\left|g(s)\right|\frac{ds}{s}\right)\left|f'\left(t\right)\right|dt\end{array}\right]$$
$$\leq \frac{\left\|g\right\|_{\infty}}{\Gamma(\alpha)}\left[\begin{array}{c}\int_a^{\sqrt{ab}}\left(\int_a^t\left(\ln\frac{s}{a}\right)^{\alpha-1}\frac{ds}{s}\right)\left|f'\left(t\right)\right|dt\\ +\int_{\sqrt{ab}}^b\left(\int_t^b\left(\ln\frac{b}{s}\right)^{\alpha-1}\frac{ds}{s}\right)\left|f'\left(t\right)\right|dt\end{array}\right].$$

Setting $t=a^{1-u}b^{u}$ and $dt=a^{1-u}b^{u}\ln\left(\frac{b}{a}\right)du$ gives

$$\left|f\left(\sqrt{ab}\right)\left[J^{\alpha}_{\sqrt{ab}-}g\left(a\right)+J^{\alpha}_{\sqrt{ab}+}g\left(b\right)\right]-\left[J^{\alpha}_{\sqrt{ab}-}\left(fg\right)\left(a\right)+J^{\alpha}_{\sqrt{ab}+}\left(fg\right)\left(b\right)\right]\right|$$
$$\leq \frac{\left\|g\right\|_{\infty}}{\Gamma(\alpha)}\left[\begin{array}{c}\int_0^{\frac{1}{2}}\left(\int_a^{a^{1-u}b^{u}}\left(\ln\frac{s}{a}\right)^{\alpha-1}\frac{ds}{s}\right)\left|f'\left(a^{1-u}b^{u}\right)\right|\left(a^{1-u}b^{u}\right)\ln\left(\frac{b}{a}\right)du\\ +\int_{\frac{1}{2}}^1\left(\int_{a^{1-u}b^{u}}^b\left(\ln\frac{b}{s}\right)^{\alpha-1}\frac{ds}{s}\right)\left|f'\left(a^{1-u}b^{u}\right)\right|\left(a^{1-u}b^{u}\right)\ln\left(\frac{b}{a}\right)du\end{array}\right]$$
$$=\frac{\left\|g\right\|_{\infty}\ln\left(\frac{b}{a}\right)}{\Gamma(\alpha)}\left[\begin{array}{c}\int_0^{\frac{1}{2}}\left(\int_a^{a^{1-u}b^{u}}\left(\ln\frac{s}{a}\right)^{\alpha-1}\frac{ds}{s}\right)\left|f'\left(a^{1-u}b^{u}\right)\right|\left(a^{1-u}b^{u}\right)du\\ +\int_{\frac{1}{2}}^1\left(\int_{a^{1-u}b^{u}}^b\left(\ln\frac{b}{s}\right)^{\alpha-1}\frac{ds}{s}\right)\left|f'\left(a^{1-u}b^{u}\right)\right|\left(a^{1-u}b^{u}\right)du\end{array}\right].$$

Using power mean inequality, we have

$$\left|f\left(\sqrt{ab}\right)\left[J^{\alpha}_{\sqrt{ab}-}g\left(a\right)+J^{\alpha}_{\sqrt{ab}+}g\left(b\right)\right]-\left[J^{\alpha}_{\sqrt{ab}-}\left(fg\right)\left(a\right)+J^{\alpha}_{\sqrt{ab}+}\left(fg\right)\left(b\right)\right]\right| \tag{2.48}$$
$$\leq \frac{\left\|g\right\|_{\infty}\ln\left(\frac{b}{a}\right)}{\Gamma(\alpha)}\left[\begin{array}{c}\left[\int_0^{\frac{1}{2}}\left(\int_a^{a^{1-u}b^{u}}\left(\ln\frac{s}{a}\right)^{\alpha-1}\frac{ds}{s}\right)du\right]^{1-\frac{1}{q}}\\ \times\left[\int_0^{\frac{1}{2}}\left(\int_a^{a^{1-u}b^{u}}\left(\ln\frac{s}{a}\right)^{\alpha-1}\frac{ds}{s}\right)\left|f'\left(a^{1-u}b^{u}\right)\right|^q\left(a^{1-u}b^{u}\right)^q du\right]^{\frac{1}{q}}\\ +\left[\int_{\frac{1}{2}}^1\left(\int_{a^{1-u}b^{u}}^b\left(\ln\frac{b}{s}\right)^{\alpha-1}\frac{ds}{s}\right)du\right]^{1-\frac{1}{q}}\\ \times\left[\int_{\frac{1}{2}}^1\left(\int_{a^{1-u}b^{u}}^b\left(\ln\frac{b}{s}\right)^{\alpha-1}\frac{ds}{s}\right)\left|f'\left(a^{1-u}b^{u}\right)\right|^q\left(a^{1-u}b^{u}\right)^q du\right]^{\frac{1}{q}}\end{array}\right]$$

$$= \frac{\|g\|_{\infty} \ln^{\alpha+1}\left(\frac{b}{a}\right)}{2^{(\alpha+1)\left(1-\frac{1}{q}\right)} (\alpha+1)^{1-\frac{1}{q}} \Gamma(\alpha+1)} \left[\begin{array}{c} \left[\int_0^{\frac{1}{2}} u^{\alpha} \left|f'\left(a^{1-u}b^u\right)\right|^q \left(a^{1-u}b^u\right)^q du\right]^{\frac{1}{q}} \\ + \left[\int_{\frac{1}{2}}^{1} (1-u)^{\alpha} \left|f'\left(a^{1-u}b^u\right)\right|^q \left(a^{1-u}b^u\right)^q du\right]^{\frac{1}{q}} \end{array} \right].$$

Using GA-convexity of $|f'|^q$ on $[a,b]$, we have

$$\left|f'(a^{1-u}b^u)\right|^q \leq (1-u)\left|f'(a)\right|^q + u\left|f'(b)\right|^q, \tag{2.49}$$

for all $u \in [0,1]$. Iif we use (2.49) in (2.48), we have

$$\left| f\left(\sqrt{ab}\right)\left[J^{\alpha}_{\sqrt{ab}-} g(a) + J^{\alpha}_{\sqrt{ab}+} g(b)\right] - \left[J^{\alpha}_{\sqrt{ab}-}(fg)(a) + J^{\alpha}_{\sqrt{ab}+}(fg)(b)\right]\right|$$

$$\leq \frac{\|g\|_{\infty} \ln^{\alpha+1}\left(\frac{b}{a}\right)}{2^{(\alpha+1)\left(1-\frac{1}{q}\right)} (\alpha+1)^{1-\frac{1}{q}} \Gamma(\alpha+1)}$$

$$\times \left[\begin{array}{c} \left[\int_0^{\frac{1}{2}} u^{\alpha} \left[(1-u)\left|f'(a)\right|^q + u\left|f'(b)\right|^q\right] \left(a^{1-u}b^u\right)^q du\right]^{\frac{1}{q}} \\ + \left[\int_{\frac{1}{2}}^{1} (1-u)^{\alpha} \left[(1-u)\left|f'(a)\right|^q + u\left|f'(b)\right|^q\right] \left(a^{1-u}b^u\right)^q du\right]^{\frac{1}{q}} \end{array} \right]$$

$$= \frac{\|g\|_{\infty} \ln^{\alpha+1}\left(\frac{b}{a}\right)}{2^{(\alpha+1)\left(1-\frac{1}{q}\right)} (\alpha+1)^{1-\frac{1}{q}} \Gamma(\alpha+1)}$$

$$\times \left[\begin{array}{c} \left[\left(\int_0^{\frac{1}{2}} u^{\alpha}(1-u)\left(a^{1-u}b^u\right)^q du\right)\left|f'(a)\right|^q + \left(\int_0^{\frac{1}{2}} u^{\alpha+1}\left(a^{1-u}b^u\right)^q du\right)\left|f'(b)\right|^q\right]^{\frac{1}{q}} \\ + \left[\left(\int_0^{\frac{1}{2}} u^{\alpha+1}\left(a^{u}b^{1-u}\right)^q du\right)\left|f'(a)\right|^q + \left(\int_0^{\frac{1}{2}} u^{\alpha}(1-u)\left(a^{u}b^{1-u}\right)^q du\right)\left|f'(b)\right|^q\right]^{\frac{1}{q}} \end{array} \right].$$

This completes the proof. □

Corollary 2.5. *In Theorem 2.8; if one takes $\alpha = 1$, $g(x) = 1$ and $\alpha = 1 = g(x)$ respectively, one has the following inequalities:*

$$\left| f\left(\sqrt{ab}\right)\int_a^b \frac{g(x)}{x}dx - \int_a^b \frac{f(x)g(x)}{x}dx\right| \tag{2.50}$$

$$\leq \frac{\|g\|_{\infty} \ln^2\left(\frac{b}{a}\right)}{2^{3\left(1-\frac{1}{q}\right)}} \left[\begin{array}{c} \left[T_3(1)\left|f'(a)\right|^q + T_4(1)\left|f'(b)\right|^q\right]^{\frac{1}{q}} \\ + \left[T_5(1)\left|f'(a)\right|^q + T_6(1)\left|f'(b)\right|^q\right]^{\frac{1}{q}} \end{array} \right],$$

$$\left| f\left(\sqrt{ab}\right) - \frac{\Gamma(\alpha+1)}{2^{1-\alpha}\left(\ln\frac{b}{a}\right)^{\alpha}}\left[J^{\alpha}_{\sqrt{ab}-} f(a) + J^{\alpha}_{\sqrt{ab}+} f(b)\right]\right| \tag{2.51}$$

$$\leq \frac{\ln\left(\frac{b}{a}\right)}{2^{(\alpha+1)\left(1-\frac{1}{q}\right)+(1-\alpha)} (\alpha+1)^{1-\frac{1}{q}}} \left[\begin{array}{c} \left[T_3(\alpha)\left|f'(a)\right|^q + T_4(\alpha)\left|f'(b)\right|^q\right]^{\frac{1}{q}} \\ + \left[T_5(\alpha)\left|f'(a)\right|^q + T_6(\alpha)\left|f'(b)\right|^q\right]^{\frac{1}{q}} \end{array} \right],$$

$$\left| f\left(\sqrt{ab}\right) - \frac{1}{\ln\frac{b}{a}}\int_a^b \frac{f(x)}{x}dx\right| \leq \frac{\ln\left(\frac{b}{a}\right)}{2^{3\left(1-\frac{1}{q}\right)}} \left[\begin{array}{c} \left[T_3(1)\left|f'(a)\right|^q + T_4(1)\left|f'(b)\right|^q\right]^{\frac{1}{q}} \\ + \left[T_5(1)\left|f'(a)\right|^q + T_6(1)\left|f'(b)\right|^q\right]^{\frac{1}{q}} \end{array} \right]. \tag{2.52}$$

Theorem 2.9. *Let $f : I \subseteq \mathbb{R}_+ \to \mathbb{R}$ be a differentiable mapping on I° and $f' \in L[a,b]$ where $a, b \in I$ with $a < b$ and $\alpha > 0$. If $|f'|^q, q > 1$, is GA-convex on $[a,b]$, $1/p + 1/q = 1$ and $g : [a,b] \to \mathbb{R}$ is continuous and geometrically symmetric with respect to $\sqrt{ab}$, then the following inequalities for fractional integrals hold:*

$$\left| f\left(\sqrt{ab}\right)\left[J^{\alpha}_{\sqrt{ab}-}g(a) + J^{\alpha}_{\sqrt{ab}+}g(b)\right] - \left[J^{\alpha}_{\sqrt{ab}-}(fg)(a) + J^{\alpha}_{\sqrt{ab}+}(fg)(b)\right]\right| \tag{2.53}$$

$$\leq \frac{\|g\|_\infty \ln^{\alpha+1}\left(\frac{b}{a}\right)}{2^{\frac{\alpha p+1}{\alpha}}(\alpha p+1)^{\frac{1}{p}}\Gamma(\alpha+1)}\left[\begin{array}{c}\left(T_7|f'(a)|^q + T_8|f'(b)|^q\right)^{\frac{1}{q}} \\ +\left(T_9|f'(a)|^q + T_{10}|f'(b)|^q\right)^{\frac{1}{q}}\end{array}\right],$$

where

$$T_7 = \int_0^{\frac{1}{2}}(1-u)\left(a^{1-u}b^u\right)^q du, \quad T_8 = \int_0^{\frac{1}{2}} u\left(a^{1-u}b^u\right)^q du,$$

$$T_9 = \int_{\frac{1}{2}}^1 (1-u)\left(a^{1-u}b^u\right)^q du, \quad T_{10} = \int_{\frac{1}{2}}^1 u\left(a^{1-u}b^u\right)^q du,$$

with $\alpha > 0$.

Proof. Using Lemma 2.2, setting $t = a^{1-u}b^u$ and $dt = a^{1-u}b^u \ln\left(\frac{b}{a}\right)du$, Hölder's inequality and (2.49) we have

$$\left| f\left(\sqrt{ab}\right)\left[J^{\alpha}_{\sqrt{ab}-}g(a) + J^{\alpha}_{\sqrt{ab}+}g(b)\right] - \left[J^{\alpha}_{\sqrt{ab}-}(fg)(a) + J^{\alpha}_{\sqrt{ab}+}(fg)(b)\right]\right|$$

$$\leq \frac{1}{\Gamma(\alpha)}\left[\begin{array}{c}\int_a^{\sqrt{ab}}\left(\int_a^t \left(\ln\frac{s}{a}\right)^{\alpha-1}|g(s)|\frac{ds}{s}\right)|f'(t)|dt \\ +\int_{\sqrt{ab}}^b\left(\int_t^b\left(\ln\frac{b}{s}\right)^{\alpha-1}|g(s)|\frac{ds}{s}\right)|f'(t)|dt\end{array}\right]$$

$$\leq \frac{\|g\|_\infty}{\Gamma(\alpha)}\left[\begin{array}{c}\int_a^{\sqrt{ab}}\left(\int_a^t \left(\ln\frac{s}{a}\right)^{\alpha-1}\frac{ds}{s}\right)|f'(t)|dt \\ +\int_{\sqrt{ab}}^b\left(\int_t^b\left(\ln\frac{b}{s}\right)^{\alpha-1}\frac{ds}{s}\right)|f'(t)|dt\end{array}\right]$$

$$= \frac{\|g\|_\infty}{\Gamma(\alpha)}\left[\begin{array}{c}\int_0^{\frac{1}{2}}\left(\int_a^{a^{1-u}b^u}\left(\ln\frac{s}{a}\right)^{\alpha-1}\frac{ds}{s}\right)\left|f'\left(a^{1-u}b^u\right)\right|\left(a^{1-u}b^u\right)\ln\left(\frac{b}{a}\right)du \\ +\int_{\frac{1}{2}}^1\left(\int_{a^{1-u}b^u}^b\left(\ln\frac{b}{s}\right)^{\alpha-1}\frac{ds}{s}\right)\left|f'\left(a^{1-u}b^u\right)\right|\left(a^{1-u}b^u\right)\ln\left(\frac{b}{a}\right)du\end{array}\right]$$

$$= \frac{\|g\|_\infty \ln\left(\frac{b}{a}\right)}{\Gamma(\alpha)}\left[\begin{array}{c}\int_0^{\frac{1}{2}}\left(\int_a^{a^{1-u}b^u}\left(\ln\frac{s}{a}\right)^{\alpha-1}\frac{ds}{s}\right)\left|f'\left(a^{1-u}b^u\right)\right|\left(a^{1-u}b^u\right)du \\ +\int_{\frac{1}{2}}^1\left(\int_{a^{1-u}b^u}^b\left(\ln\frac{b}{s}\right)^{\alpha-1}\frac{ds}{s}\right)\left|f'\left(a^{1-u}b^u\right)\right|\left(a^{1-u}b^u\right)du\end{array}\right]$$

$$\leq \frac{\|g\|_\infty \ln\left(\frac{b}{a}\right)}{\Gamma(\alpha)}\left[\begin{array}{c}\left(\int_0^{\frac{1}{2}}\left(\int_a^{a^{1-u}b^u}\left(\ln\frac{s}{a}\right)^{\alpha-1}\frac{ds}{s}\right)^p du\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\left|f'\left(a^{1-u}b^u\right)\right|^q\left(a^{1-u}b^u\right)^q du\right)^{\frac{1}{q}} \\ +\left(\int_{\frac{1}{2}}^1\left(\int_{a^{1-u}b^u}^b\left(\ln\frac{b}{s}\right)^{\alpha-1}\frac{ds}{s}\right)^p du\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^1\left|f'\left(a^{1-u}b^u\right)\right|^q\left(a^{1-u}b^u\right)^q du\right)^{\frac{1}{q}}\end{array}\right]$$

$$= \frac{\|g\|_\infty \ln^{\alpha+1}\left(\frac{b}{a}\right)}{2^{\frac{\alpha p+1}{p}}(\alpha p+1)^{\frac{1}{p}}\Gamma(\alpha+1)}\left[\begin{array}{c}\left(\int_0^{\frac{1}{2}}\left|f'\left(a^{1-u}b^u\right)\right|^q\left(a^{1-u}b^u\right)^q du\right)^{\frac{1}{q}} \\ +\left(\int_{\frac{1}{2}}^1\left|f'\left(a^{1-u}b^u\right)\right|^q\left(a^{1-u}b^u\right)^q du\right)^{\frac{1}{q}}\end{array}\right]$$

$$\leq \frac{\|g\|_{\infty}\ln^{\alpha+1}\left(\frac{b}{a}\right)}{2^{\frac{\alpha p+1}{p}}(\alpha p+1)^{\frac{1}{p}}\Gamma(\alpha+1)}\left[\begin{array}{c}\left(\int_0^{\frac{1}{2}}\left[(1-u)\left|f'(a)\right|^q+u\left|f'(b)\right|^q\right]\left(a^{1-u}b^u\right)^q du\right)^{\frac{1}{q}}\\ +\left(\int_{\frac{1}{2}}^{1}\left[(1-u)\left|f'(a)\right|^q+u\left|f'(b)\right|^q\right]\left(a^{1-u}b^u\right)^q du\right)^{\frac{1}{q}}\end{array}\right]$$

$$=\frac{\|g\|_{\infty}\ln^{\alpha+1}\left(\frac{b}{a}\right)}{2^{\frac{\alpha p+1}{p}}(\alpha p+1)^{\frac{1}{p}}\Gamma(\alpha+1)}$$

$$\times\left[\begin{array}{c}\left(\left[\int_0^{\frac{1}{2}}(1-u)\left(a^{1-u}b^u\right)^q du\right]\left|f'(a)\right|^q+\left[\int_0^{\frac{1}{2}}u\left(a^{1-u}b^u\right)^q du\right]\left|f'(b)\right|^q\right)^{\frac{1}{q}}\\ +\left(\left[\int_{\frac{1}{2}}^{1}(1-u)\left(a^{1-u}b^u\right)^q du\right]\left|f'(a)\right|^q+\left[\int_{\frac{1}{2}}^{1}u\left(a^{1-u}b^u\right)^q du\right]\left|f'(b)\right|^q\right)^{\frac{1}{q}}\end{array}\right].$$

This completes the proof. □

Corollary 2.6. *In Theorem 2.9; if one takes $\alpha=1$, $g(x)=1$ and $\alpha=1=g(x)$ respectively, one has the following inequalities:*

$$\left|f\left(\sqrt{ab}\right)\int_a^b\frac{g(x)}{x}dx-\int_a^b\frac{f(x)g(x)}{x}dx\right|$$

$$\leq\frac{\|g\|_{\infty}\ln^2\left(\frac{b}{a}\right)}{2^{1+\frac{1}{p}}(p+1)^{\frac{1}{p}}}\left[\begin{array}{c}\left(T_7\left|f'(a)\right|^q+T_8\left|f'(b)\right|^q\right)^{\frac{1}{q}}\\ +\left(T_9\left|f'(a)\right|^q+T_{10}\left|f'(b)\right|^q\right)^{\frac{1}{q}}\end{array}\right], \tag{2.54}$$

$$\left|f\left(\sqrt{ab}\right)-\frac{\Gamma(\alpha+1)}{2^{1-\alpha}\left(\ln\frac{b}{a}\right)^{\alpha}}\left[J^{\alpha}_{\sqrt{ab}-}f(a)+J^{\alpha}_{\sqrt{ab}+}f(b)\right]\right| \tag{2.55}$$

$$\leq\frac{\ln\left(\frac{b}{a}\right)}{2^{\left(\frac{\alpha p+1}{p}\right)+1-\alpha}(\alpha p+1)^{\frac{1}{p}}}\left[\begin{array}{c}\left(T_7\left|f'(a)\right|^q+T_8\left|f'(b)\right|^q\right)^{\frac{1}{q}}\\ +\left(T_9\left|f'(a)\right|^q+T_{10}\left|f'(b)\right|^q\right)^{\frac{1}{q}}\end{array}\right],$$

$$\left|f\left(\sqrt{ab}\right)-\frac{1}{\ln\frac{b}{a}}\int_a^b\frac{f(x)}{x}dx\right|\leq\frac{\ln\left(\frac{b}{a}\right)}{2^{1+\frac{1}{p}}(p+1)^{\frac{1}{p}}}\left[\begin{array}{c}\left(T_7\left|f'(a)\right|^q+T_8\left|f'(b)\right|^q\right)^{\frac{1}{q}}\\ +\left(T_9\left|f'(a)\right|^q+T_{10}\left|f'(b)\right|^q\right)^{\frac{1}{q}}\end{array}\right]. \tag{2.56}$$

References

[1] İ. İşcan, *Generalization of different type integral inequalities for s-convex functions via fractional integrals,* Applicable Anal., **93**(9) (2014), 1846-1862.

[2] İ. İşcan, *New general integral inequalities for quasi-geometrically convex functions via fractional integrals,* J. Inequal. Appl., **2013**(491) (2013), 15 pp.

[3] İ. İşcan, *On generalization of different type integral inequalities for s-convex functions via fractional integrals,* Math. Sci. Appl. E-Notes, **2**(1) (2014), 55-67.

[4] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and applications of fractional differential equations*, Elsevier, Amsterdam, 2006.

[5] M. A. Latif, S. S. Dragomir and E. Momaniat, *Some Fejer type integral inequalities for geometrically-arithmetically-convex functions with applications,* RGMIA Research Report Collection, **18**(2015), Article 25, 13 pp.

[6] C. P. Niculescu, *Convexity according to the geometric mean,* Math. Inequal. Appl. **3**(2) (2000), 155-167.

[7] C. P. Niculescu, *Convexity according to means,* Math. Inequal. Appl. **6**(4) (2003), 571-579.

[8] A.P. Prudnikov, Y.A. Brychkov and O.I. Marichev, *Integral and series, In: Elementary Functions*, vol. 1. Nauka, Moscow, 1981.

[9] J. Wang, X. Li, M. Fečkan and Y. Zhou, *Hermite-Hadamard-type inequalities for Riemann-Liouville fractional integrals via two kinds of convexity,* Appl. Anal., **92**(11) (2012), 2241-2253.

[10] J. Wang, C. Zhu and Y. Zhou, *New generalized Hermite-Hadamard type inequalities and applications to special means,* J. Inequal. Appl., **2013**(325) (2013), 15 pp.

[1]Department of Mathematics,
Karadeniz Technical University,
61080, Trabzon, Turkey
E-mail address: mkunt@ktu.edu.tr

[2]Department of Mathematics,
Giresun University,
28200, Giresun, Turkey
E-mail address: imdat.iscan@giresun.edu.tr; imdati@yahoo.com