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PARAMETERIZED HERMITE-HADAMARD TYPE INEQUALITIES FOR FRACTIONAL INTEGRALS

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Abstract. The paper presents Hermite-Hadamard type inequalities, which involve Riemann-Liouville fractional integrals and contain an arbitrary parameter from the interval of definition of twice differentiable convex and concave functions.

1. Introduction

Fractional calculus is the notion of integrals and derivatives of arbitrary order, which is the generalization of integer-order differentiation and n-fold integration. The beginning of fractional calculus is considered to be the correspondence between L'Hospital and Leibniz in 1695, where the idea for differentiation of non-integer orders was discussed [9]. This correspondence gave birth to the idea of fractional calculus. Further contributions in this area were made by Euler, Laplace, Fourier, Abel, Liouville, Riemann, Grunwald, Letnikov, Hadamard, Weyl, Riesz, Marchaud, Kober and Caputo [2, 9, 20–22]. Fractional calculus plays an important role in various fields such as Electricity, Biology, Economics, Signal and Image Processing.
The following definition is well-known in the literature and is widely used:

Definition 1.1. A function $\zeta : I \to \mathbb{R}$, defined on the interval I in $\mathbb{R}$, is said to be convex on I if

$$\zeta(\rho\tau_1 + (1-\rho)\tau_2) \leq \rho\zeta(\tau_1) + (1-\rho)\zeta(\tau_2), \tag{1.1}$$

for all $\tau_1, \tau_2 \in I$ and $0 \leq \rho \leq 1$. Also we say that ζ is concave on I, if the inequality given in (1.1) holds in the reverse direction.

Corresponding to the definition of convex functions the following double inequality has played a very important role in various fields of science.

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Theorem 1.1. *Let ζ be a convex function. Then for $\tau_1, \tau_2 \in I$ with $\tau_1 < \tau_2$, $\xi \in [\tau_1, \tau_2]$ we have:*

$$\zeta\left(\frac{\tau_1+\tau_2}{2}\right) \leq \frac{1}{\tau_2-\tau_1}\int_{\tau_1}^{\tau_2} \zeta(\xi)d\xi \leq \frac{\zeta(\tau_1)+\zeta(\tau_2)}{2}. \tag{1.2}$$

The order of inequality in (1.2) is reversed if ζ is concave. This inequality is the widely used Hermite-Hadamard inequality, which gives an estimate from both sides of the mean i.e. from above and below of the mean value of a convex function and ensures the integrability of any convex function too. For more information about the Hermite-Hadamard inequality, the interested readers can see [1, 4–8, 12–15, 23, 24].

We recall the definition of Riemann-Liouville (R-L) fractional integrals which we will use in further results:

Definition 1.2 ([9])**.** Let $\zeta \in L_1[\tau_1, \tau_2]$ with $\tau_1 \geq 0$. The Riemann-Liouville (R-L) fractional integral operators $J^{\eta}_{\tau_1^+}\zeta$ and $J^{\eta}_{\tau_2^-}\zeta$ of order $\eta > 0$ are defined by:

$$J^{\eta}_{\tau_1^+}\zeta(\xi) = \frac{1}{\Gamma(\eta)}\int_{\tau_1}^{\xi} (\xi-\rho)^{\eta-1}\zeta(\rho)d\rho, \quad \text{with} \quad \xi > \tau_1$$

and

$$J^{\eta}_{\tau_2^-}\zeta(\xi) = \frac{1}{\Gamma(\eta)}\int_{\xi}^{\tau_2} (\rho-\xi)^{\eta-1}\zeta(\rho)d\rho, \quad \text{with} \quad \xi < \tau_2.$$

Here, $\Gamma(\eta)$ represents the Gamma function given by:

$$\Gamma(\eta) = \int_{0}^{\infty} e^{-u}u^{\eta-1}du.$$

Here $J^{0}_{\tau_1^+}\zeta(\xi) = J^{0}_{\tau_2^-}\zeta(\xi) = \zeta(\xi)$. When $\eta = 1$, the R-L fractional integrals reduce to Riemann integrals.

E. Set [10] firstly examined Ostrowski type inequalities involving R-L fractional integrals. Also, Sarikaya et al. [12] studied the fractional form of the inequality (1.2). For other of-late applications of fractional derivatives and fractional integrals, one can see [3, 4, 8, 11, 13–19]. The fractional form of Hermite-Hadamard inequality is given below. We will design new bounds for the difference of rightmost terms in this inequality.

Theorem 1.2 ([12])**.** *Let $\zeta : [\tau_1, \tau_2] \to \mathbb{R}$ be a positive function with $0 \leq \tau_1 < \tau_2$ and $\zeta \in L_1[\tau_1, \tau_2]$. If ζ is convex function on $[\tau_1, \tau_2]$, then the following inequality for R-L fractional integrals holds:*

$$\zeta\left(\frac{\tau_1+\tau_2}{2}\right) \leq \frac{\Gamma(\eta+1)}{2(\tau_2-\tau_1)^{\eta}}\left[J^{\eta}_{\tau_1+}\zeta(\tau_2) + J^{\eta}_{\tau_2-}\zeta(\tau_1)\right] \leq \frac{\zeta(\tau_1)+\zeta(\tau_2)}{2}. \tag{1.3}$$

Remark 1.1. By replacing $\eta = 1$ in (1.3), we get the inequality (1.2).

The following result is due to Sarikaya et al. [12], which contains a differentiable convex function. In this result the difference of rightmost terms in inequality (1.3) was bounded.

Theorem 1.3 ([12]). *Let $\zeta : I \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on I° (the interior of I) such that $\tau_1, \tau_2 \in I^\circ$ with $\tau_1 < \tau_2$. If $|\zeta'|$ is convex function on $[\tau_1, \tau_2]$, then the following inequality for fractional integrals holds:*

$$\left|\frac{\zeta(\tau_1)+\zeta(\tau_2)}{2} - \frac{\Gamma(\eta+1)}{2(\tau_2-\tau_1)^\eta}\left[J^\eta_{\tau_1^+}\zeta(\tau_2) + J^\eta_{\tau_2^-}\zeta(\tau_1)\right]\right| \le \frac{(\tau_2-\tau_1)}{2(\eta+1)}\left(1-\frac{1}{2^\eta}\right)\left[\zeta'(\tau_1)+\zeta'(\tau_2)\right].$$

In [7], Yu ming chu et al. have discovered an integral identity involving R-L fractional integrals, which is given below:

Lemma 1.1. *Let $\zeta : I \subset [0,\infty) \to \mathbb{R}$ be a twice differentiable function on I°, such that τ_1, $\tau_2 \in I^\circ$ with $\tau_1 < \tau_2$. If $\zeta'' \in L_1[\tau_1, \tau_2]$, then for $\eta > 0$, $\theta \in [\tau_1, \tau_2]$ the following identity holds:*

$$\begin{aligned}
&\frac{\zeta'(\theta)\left((\theta-\tau_1)^{\eta+1}-(\tau_2-\theta)^{\eta+1}\right)+(\eta+1)\zeta(\tau_2)(\tau_2-\theta)^\eta+(\eta+1)\zeta(\tau_1)(\theta-\tau_1)^\eta}{(\eta+1)(\tau_2-\tau_1)}\\
&-\frac{\Gamma(\eta+1)}{(\tau_2-\tau_1)}\left[J^\eta_{\tau_1^+}\zeta(\theta)+J^\eta_{\tau_2^-}\zeta(\theta)\right] = \frac{(\theta-\tau_1)^{\eta+2}}{(\eta+1)(\tau_2-\tau_1)}\int_0^1\left(1-\rho^{\eta+1}\right)\zeta''(\rho\tau_1+(1-\rho)\theta)d\rho\\
&+\frac{(\tau_2-\theta)^{\eta+2}}{(\eta+1)(\tau_2-\tau_1)}\int_0^1\left(1-\rho^{\eta+1}\right)\zeta''(\rho\tau_2+(1-\rho)\theta)d\rho.
\end{aligned} \tag{1.4}$$

Here, in the current paper, we deduce some parameterized inequalities of Hermite-Hadamard type via R-L fractional integrals (see Theorems 2.1,2.2,2.3,2.5,2.4). The novelty of these results is that they contain an arbitrary parameter θ from the interval of definition of twice differentiable convex or concave functions. Further more when the parameter θ is replaced by the midpoint of the interval we get different bounds for the trapezoidal formula.

2. Main results

Before giving our main results we introduce some notations for the sake of simplifications. Let $\zeta : I \subset [0,\infty) \to \mathbb{R}$ be a twice differentiable function on I° and $\tau_1, \tau_2 \in I^\circ$ with $\tau_1 < \tau_2$. If $\zeta'' \in L_1[\tau_1, \tau_2]$ ($|\zeta''|$ is integrable on $[\tau_1, \tau_2]$), then for all $\theta \in [\tau_1, \tau_2]$ and $\eta > 0$, we define L_ζ by:

$$\begin{aligned}
&L_\zeta(\theta,\eta,\tau_1,\tau_2)\\
=\;&\frac{\zeta'(\theta)\left((\theta-\tau_1)^{\eta+1}-(\tau_2-\theta)^{\eta+1}\right)+(\eta+1)\zeta(\tau_2)(\tau_2-\theta)^\eta+(\eta+1)\zeta(\tau_1)(\theta-\tau_1)^\eta}{(\eta+1)(\tau_2-\tau_1)}\\
&-\frac{\Gamma(\eta+1)}{(\tau_2-\tau_1)}\left[J^\eta_{\tau_1^+}\zeta(\theta)+J^\eta_{\tau_2^-}\zeta(\theta)\right].
\end{aligned}$$

For $\theta = \frac{\tau_1+\tau_2}{2}$, we have

$$L_\zeta\left(\frac{\tau_1+\tau_2}{2},\eta,\tau_1,\tau_2\right) = \left(\frac{\tau_2-\tau_1}{2}\right)^{\eta-1}\frac{\zeta(\tau_1)+\zeta(\tau_2)}{2} - \frac{\Gamma(\eta+1)}{(\tau_2-\tau_1)}\left[J^\eta_{\tau_1^+}\zeta\left(\frac{\tau_1+\tau_2}{2}\right) + J^\eta_{\tau_2^-}\zeta\left(\frac{\tau_1+\tau_2}{2}\right)\right],$$

and by putting $\eta = 1$ in this, we get

$$L_\zeta\left(\frac{\tau_1+\tau_2}{2},1,\tau_1,\tau_2\right) = \frac{\zeta(\tau_1)+\zeta(\tau_2)}{2} - \frac{1}{\tau_2-\tau_1}\int_{\tau_1}^{\tau_2}\zeta(\theta)d\theta,$$

which is the difference between the two right most terms in (1.2) or the celebrated trapezoidal formula term.
Now we find out our first parameterized bound.

Theorem 2.1. *Let all the requisites of Lemma* 1.1 *hold. Additionally, if* $|\zeta''|$ *is convex function on* $[\tau_1,\tau_2]$*, then we have:*

$$|L_\zeta(\theta,\eta,\tau_1,\tau_2)| \leq \frac{(\theta-\tau_1)^{\eta+2}\left[|\zeta''(\tau_1)| + \frac{\eta+4}{\eta+2}|\zeta''(\theta)|\right] + (\tau_2-\theta)^{\eta+2}\left[\zeta''(\tau_2)| + \frac{\eta+4}{\eta+2}|\zeta''(\theta)|\right]}{2(\tau_2-\tau_1)(\eta+3)}. \tag{2.1}$$

Proof. Using Lemma 1.1, well known triangle inequality and convexity of $|\zeta''|$, we have

$$\begin{aligned}
|L_\zeta(\theta,\eta,\tau_1,\tau_2)| &\leq \frac{(\theta-\tau_1)^{\eta+2}}{(\eta+1)(\tau_2-\tau_1)}\int_0^1(1-\rho^{\eta+1})|\zeta''(\rho\tau_1+(1-\rho)\theta)|d\rho \\
&+\frac{(\tau_2-\theta)^{\eta+2}}{(\eta+1)(\tau_2-\tau_1)}\int_0^1(1-\rho^{\eta+1})|\zeta''(\rho\tau_2+(1-\rho)\theta)|d\rho \\
&\leq \frac{(\theta-\tau_1)^{\eta+2}}{(\eta+1)(\tau_2-\tau_1)}\int_0^1\left(1-\rho^{\eta+1}\right)\left[\rho|\zeta''(\tau_1)| + (1-\rho)|\zeta''(\theta)|\right]d\rho \\
&+\frac{(\tau_2-\theta)^{\eta+2}}{(\eta+1)(\tau_2-\tau_1)}\int_0^1\left(1-\rho^{\eta+1}\right)\left[\rho|\zeta''(\tau_2)| + (1-\rho)|\zeta''(\theta)|\right]d\rho \\
&= \frac{(\theta-\tau_1)^{\eta+2}}{(\eta+1)(\tau_2-\tau_1)}\left[\left(\frac{\eta+1}{2(\eta+3)}\right)|\zeta''(\tau_1)| + \left(\frac{(\eta+4)(\eta+1)}{2(\eta+3)(\eta+2)}\right)|\zeta''(\theta)|\right] \\
&+ \frac{(\tau_2-\theta)^{\eta+2}}{(\eta+1)(\tau_2-\tau_1)}\left[\left(\frac{\eta+1}{2(\eta+3)}\right)|\zeta''(\tau_2)| + \left(\frac{(\eta+4)(\eta+1)}{2(\eta+3)(\eta+2)}\right)|\zeta''(\theta)|\right] \\
&= \frac{(\theta-\tau_1)^{\eta+2}\left[|\zeta''(\tau_1)| + \frac{\eta+4}{\eta+2}|\zeta''(\theta)|\right] + (\tau_2-\theta)^{\eta+2}\left[\zeta''(\tau_2)| + \frac{\eta+4}{\eta+2}|\zeta''(\theta)|\right]}{2(\tau_2-\tau_1)(\eta+3)}.
\end{aligned}$$

The proof is completed. □

Corollary 2.1. *Under the hypothesis of Theorem* 2.1, *we get*

$$\left|L_\zeta\left(\frac{\tau_1+\tau_2}{2},\eta,\tau_1,\tau_2\right)\right| \le \left(\frac{\tau_2-\tau_1}{2}\right)^{\eta+1}\frac{|\zeta''(\tau_1)|+|\zeta''(\tau_2)|}{2(\eta+2)}. \tag{2.2}$$

Proof. If we put $\theta=\frac{\tau_1+\tau_2}{2}$ in inequality (2.1) and use the convexity of $|\zeta''|$, we get the desired inequality. □

The associated versions for powers of the second derivative absolute values of the function are included in the following theorems.

Theorem 2.2. *Let all the requisites of Lemma* 1.1 *hold. Additionally, if* $|\zeta''|^s$ *is convex function on* $[\tau_1,\tau_2]$ *for* $s>1$ *and* $r^{-1}+s^{-1}=1$, *then we have the following inequality:*

$$\begin{aligned}|L_\zeta(\theta,\eta,\tau_1,\tau_2)| \le\ & \left(\frac{1}{2}\right)^{\frac{1}{s}}\frac{M^{\frac{1}{r}}}{(\eta+1)(\tau_2-\tau_1)}\Big[(\theta-\tau_1)^{\eta+2}\left(|\zeta''(\tau_1)|^s+|\zeta''(\theta)|^s\right)^{\frac{1}{s}} \\ & +(\tau_2-\theta)^{\eta+2}\left(|\zeta''(\tau_2)|^s+|\zeta''(\theta)|^s\right)^{\frac{1}{s}}\Big],\end{aligned} \tag{2.3}$$

where

$$M=\frac{\Gamma(1+r)\Gamma(\frac{1}{\eta+1})}{(\eta+1)\Gamma(1+r+\frac{1}{\eta+1})}.$$

Proof. Using Lemma 1.1, triangle and Hölder inequalities, we have

$$\begin{aligned}|L_\zeta(\theta,\eta,\tau_1,\tau_2)| \le\ & \frac{(\theta-\tau_1)^{\eta+2}}{(\eta+1)(\tau_2-\tau_1)}\int_0^1(1-\rho^{\eta+1})|\zeta''(\rho\tau_1+(1-\rho)\theta)|d\rho \\ & +\frac{(\tau_2-\theta)^{\eta+2}}{(\eta+1)(\tau_2-\tau_1)}\int_0^1(1-\rho^{\eta+1})|\zeta''(\rho\tau_2+(1-\rho)\theta)|d\rho \\ \le\ & \frac{(\theta-\tau_1)^{\eta+2}}{(\eta+1)(\tau_2-\tau_1)}\left(\int_0^1\left(1-\rho^{\eta+1}\right)^r d\rho\right)^{\frac{1}{r}}\left(\int_0^1|\zeta''(\rho\tau_1+(1-\rho)\theta)|^s d\rho\right)^{\frac{1}{s}} \\ & +\frac{(\tau_2-\theta)^{\eta+2}}{(\eta+1)(\tau_2-\tau_1)}\left(\int_0^1\left(1-\rho^{\eta+1}\right)^r d\rho\right)^{\frac{1}{r}}\left(\int_0^1|\zeta''(\rho\tau_2+(1-\rho)\theta)|^s d\rho\right)^{\frac{1}{s}}.\end{aligned}$$

Using the convexity of $|\zeta''|^s$, we get

$$\begin{aligned}\int_0^1 |\ \zeta''(\rho\tau_1+(1-\rho)\theta)\ |^s\ d\rho \ \le\ & \int_0^1\left(\rho\ |\ \zeta''(\tau_1)\ |^s+(1-\rho)\ |\ \zeta''(\theta)\ |^s\right)d\rho \\ =\ & \frac{|\zeta''(\tau_1)|^s+|\zeta''(\theta)|^s}{2}.\end{aligned}$$

Similarly

$$\int_0^1 |\, \zeta''(\rho\tau_2+(1-\rho)\theta)\,|^s\, d\rho \;\le\; \int_0^1 (\rho\,|\,\zeta''(\tau_2)\,|^s + (1-\rho)\,|\,\zeta''(\theta)\,|^s)\, d\rho = \frac{|\zeta''(\tau_2)|^s+|\zeta''(\theta)|^s}{2}.$$

also

$$\int_0^1 \left(1-\rho^{\eta+1}\right)^r d\rho = \frac{1}{\eta+1}\int_0^1 u^{\frac{1}{\eta+1}-1}(1-u)^r du = \frac{\Gamma(1+r)\Gamma(\frac{1}{\eta+1})}{(\eta+1)\Gamma(1+r+\frac{1}{\eta+1})} = M.$$

Combining all the above inequalities, we have the conclusion (2.3). □

Corollary 2.2. *Under the hypothesis of Theorem* 2.2, *we have*

$$\begin{aligned}\left|L_\zeta\left(\frac{\tau_1+\tau_2}{2},\eta,\tau_1,\tau_2\right)\right| \le& \left(\frac{1}{2}\right)^{\frac{1}{s}} \frac{M^{\frac{1}{r}}}{2(\eta+1)}\left(\frac{\tau_2-\tau_1}{2}\right)^{\eta+1}\\ &\times\left[\left(|\zeta''(\tau_1)|^s+\left|\zeta''\left(\frac{\tau_1+\tau_2}{2}\right)\right|^s\right)^{\frac{1}{s}}\right.\\ &\left.+\left(|\zeta''(\tau_2)|^s+\left|\zeta''\left(\frac{\tau_1+\tau_2}{2}\right)\right|^s\right)^{\frac{1}{s}}\right]\\ \le& M^{\frac{1}{r}}\left(\frac{\tau_2-\tau_1}{2}\right)^{\eta+1}\frac{|\zeta''(\tau_1)|+|\zeta''(\tau_2)|}{2(\eta+1)}. \end{aligned} \tag{2.4}$$

Proof. In inequality (2.3), if we take $\theta=\frac{\tau_1+\tau_2}{2}$, we get the first bound in (2.4). The second bound in (2.4) can be obtained by using the convexity of $|\zeta''|^s$ and the fact that:

$$\sum_{\nu=1}^{\mu}(x_\nu+y_\nu)^\omega \le \sum_{\nu=1}^{\mu} x_\nu^\omega + \sum_{\nu=1}^{\mu} y_\nu^\omega\ ,$$

for $0\le\omega\le 1$ and $x_i, y_i \ge 0$, where $i=1,2,....\mu$. □

A more general parameterized bound can be prolonged in the following Theorem.

Theorem 2.3. *Let all the requisites of Lemma* 1.1 *hold. Additionally, if* $|\zeta''|^s$ *is convex function on* $[\tau_1,\tau_2]$ *for* $s\ge 1$, *then we have the following inequality:*

$$\begin{aligned}&|L_\zeta(\theta,\eta,\tau_1,\tau_2)|\\ \le& \frac{\left(\frac{\eta+2}{\eta+1}\right)^{\frac{1}{s}}}{(\eta+2)(\tau_2-\tau_1)}\left[(\theta-\tau_1)^{\eta+2}\left(\frac{\eta+1}{2(\eta+3)}|\zeta''(\tau_1)|^s+\frac{(\eta+4)(\eta+1)}{2(\eta+3)(\eta+2)}|\zeta''(\theta)|^s\right)^{\frac{1}{s}}\right.\\ &\left.+(\tau_2-\theta)^{\eta+2}\left(\frac{\eta+1}{2(\eta+3)}|\zeta''(\tau_2)|^s+\frac{(\eta+4)(\eta+1)}{2(\eta+3)(\eta+2)}|\zeta''(\theta)|^s\right)^{\frac{1}{s}}\right]. \end{aligned} \tag{2.5}$$

Proof. Using Lemma 1.1 and Power mean-inequality, we have

$$\begin{aligned}
&|L_\zeta(\theta,\eta,\tau_1,\tau_2)| \\
&\le \frac{(\theta-\tau_1)^{\eta+2}}{(\eta+1)(\tau_2-\tau_1)}\int_0^1\left(1-\rho^{\eta+1}\right)|\zeta''(\rho\tau_1+(1-\rho)\theta)|d\rho \\
&\quad+\frac{(\tau_2-\theta)^{\eta+2}}{(\eta+1)(\tau_2-\tau_1)}\int_0^1\left(1-\rho^{\eta+1}\right)|\zeta''(\rho\tau_2+(1-\rho)\theta)|d\rho \\
&\le \frac{(\theta-\tau_1)^{\eta+2}}{(\eta+1)(\tau_2-\tau_1)}\left(\int_0^1(1-\rho^{\eta+1})d\rho\right)^{1-\frac{1}{s}}\left(\int_0^1\left(1-\rho^{\eta+1}\right)|\zeta''(\rho\tau_1+(1-\rho)\theta)|^s d\rho\right)^{\frac{1}{s}} \\
&\qquad+\frac{(\tau_2-\theta)^{\eta+2}}{(\eta+1)(\tau_2-\tau_1)}\left(\int_0^1(1-\rho^{\eta+1})d\rho\right)^{1-\frac{1}{s}}\left(\int_0^1\left(1-\rho^{\eta+1}\right)|\zeta''(\rho\tau_2+(1-\rho)\theta)|^s d\rho\right)^{\frac{1}{s}}
\end{aligned} \tag{2.6}$$

since $|\zeta''|^s$ is convex function on $[\tau_1,\tau_2]$, so we have

$$\begin{aligned}
\int_0^1(1-\rho^{\eta+1})|\zeta''(\rho\tau_1+(1-\rho)\theta)|^s d\rho &\le \int_0^1\left[\rho(1-\rho^{\eta+1})|\zeta''(\tau_1)|^s+(1-\rho)(1-\rho^{\eta+1})|\zeta''(\theta)|^s\right]d\rho \\
&= \frac{\eta+1}{2(\eta+3)}|\zeta''(\tau_1)|^s+\frac{(\eta+4)(\eta+1)}{2(\eta+3)(\eta+2)}|\zeta''(\theta)|^s,
\end{aligned} \tag{2.7}$$

similarly we can write

$$\begin{aligned}
\int_0^1(1-\rho^{\eta+1})|\zeta''(\rho\tau_2+(1-\rho)\theta)|^s d\rho &\le \int_0^1\left[\rho(1-\rho^{\eta+1})|\zeta''(\tau_2)|^s+(1-\rho)(1-\rho^{\eta+1})|\zeta''(\theta)|^s\right]d\rho \\
&= \frac{\eta+1}{2(\eta+3)}|\zeta''(\tau_2)|^s+\frac{(\eta+4)(\eta+1)}{2(\eta+3)(\eta+2)}|\zeta''(\theta)|^s.
\end{aligned} \tag{2.8}$$

Also we have

$$\left(\int_0^1(1-\rho^{\eta+1})d\rho\right)^{1-\frac{1}{s}}=\left(\frac{\eta+1}{\eta+2}\right)^{1-\frac{1}{s}}.$$

Now using (2.7) and (2.8) in (2.6), we get (2.5). □

Corollary 2.3. *Under the hypothesis of Theorem* 2.3, *we have*

$$\begin{aligned}
&\left|L_\zeta\left(\frac{\tau_1+\tau_2}{2},\eta,\tau_1,\tau_2\right)\right| \\
&\leq \frac{\left(\frac{\eta+2}{\eta+1}\right)^{\frac{1}{s}}\left(\frac{\tau_2-\tau_1}{2}\right)^{\eta+1}}{2(\eta+2)}\left[\left(\frac{\eta+1}{2(\eta+3)}|\zeta''(\tau_1)|^s+\frac{(\eta+4)(\eta+1)}{2(\eta+3)(\eta+2)}|\zeta''\left(\frac{\tau_1+\tau_2}{2}\right)|^s\right)^{\frac{1}{s}}\right. \\
&\left.+\left(\frac{\eta+1}{2(\eta+3)}|\zeta''(\tau_2)|^s+\frac{(\eta+4)(\eta+1)}{2(\eta+3)(\eta+2)}|\zeta''\left(\frac{\tau_1+\tau_2}{2}\right)|^s\right)^{\frac{1}{s}}\right] \\
&\leq \left(\frac{\tau_2-\tau_1}{2}\right)^{\eta+1}\left(\left(\frac{\eta+2}{2(\eta+3)}\right)^{\frac{1}{s}}+\left(\frac{\eta+4}{2(\eta+3)}\right)^{\frac{1}{s}}\right)\frac{|\zeta''(\tau_1)|+|\zeta''(\tau_2)|}{2(\eta+2)}. \qquad (2.9)
\end{aligned}$$

Proof. In inequality (2.5), if we take $\theta=\frac{\tau_1+\tau_2}{2}$, we get the first bound in (2.9). The second bound in (2.9) can be obtained by using the fact that:

$$\sum_{\nu=1}^{\mu}(x_\nu+y_\nu)^\omega \leq \sum_{\nu=1}^{\mu}x_\nu^\omega+\sum_{\nu=1}^{\mu}y_\nu^\omega\ ,$$

for $0\leq\omega\leq 1$ and $x_i,y_i\geq 0$, where $i=1,2,....\mu$ and then convexity of $|\zeta''|$. □

Remark 2.1. By putting $s=1$ in (2.9), we get the inequality (2.2).

Instead of convexity, using concavity property of the function we get two different inequalities, which are given below.

Theorem 2.4. *Let all the requisites of Lemma* 1.1 *hold.. Additionally, If* $|\zeta''|^s$ *is concave on* $[\tau_1,\tau_2]$ *for each* $s>1$, *then the following inequality holds:*

$$|L_\zeta(\theta,\eta,\tau_1,\tau_2)| \leq \frac{(\theta-\tau_1)^{\eta+2}\left|\zeta''\left(\frac{(\eta+2)\tau_1+(\eta+4)\theta}{2(\eta+3)}\right)\right|+(\tau_2-\theta)^{\eta+2}\left|\zeta''\left(\frac{(\eta+2)\tau_2+(\eta+4)\theta}{2(\eta+3)}\right)\right|}{(\eta+2)(\tau_2-\tau_1)}. \qquad (2.10)$$

for each θ *in* $[\tau_1,\tau_2]$.

Proof. By power mean inequality, we have

$$\begin{aligned}
(\rho|\zeta''(\tau_1)|+(1-\rho)|\zeta''(\tau_2)|)^s &\leq \rho|\zeta''(\tau_1)|^s+(1-\rho)|\zeta''(\tau_2)|^s \\
&\leq |\zeta''(\rho\tau_1+(1-\rho)\tau_2)|^s,\ (\text{ by concavity of } |\zeta''|^s\ )
\end{aligned}$$

and therefore

$$|\zeta''(\rho\tau_1+(1-\rho)\tau_2)|\geq\rho|\zeta''(\tau_1)|+(1-\rho)|\zeta''(\tau_2)|,$$

this shows that $|\zeta''|$ is also concave. Now using Lemma 1.1, triangular inequality and then Jensen integral inequality in turn, we have

$$
\begin{aligned}
|L_\zeta(\theta,\eta,\tau_1,\tau_2)| &\leq \frac{(\theta-\tau_1)^{\eta+2}}{(\eta+1)(\tau_2-\tau_1)}\int\limits_0^1 \left(1-\rho^{\eta+1}\right) \mid \zeta''(\rho\tau_1+(1-\rho)\theta) \mid d\rho \\
&\quad + \frac{(\tau_2-\theta)^{\eta+2}}{(\eta+1)(\tau_2-\tau_1)}\int\limits_0^1 \left(1-\rho^{\eta+1}\right) \mid \zeta''(\rho\tau_2+(1-\rho)\theta) \mid d\rho \\
&\leq \frac{(\theta-\tau_1)^{\eta+2}}{(\eta+1)(\tau_2-\tau_1)}\left(\int\limits_0^1 (1-\rho^{\eta+1})d\rho\right)\left|\zeta''\left(\frac{\int_0^1 (1-\rho^{\eta+1})(\rho\tau_1+(1-\rho)\theta)d\rho}{\int_0^1 (1-\rho^{\eta+1})d\rho}\right)\right| \\
&\quad + \frac{(\tau_2-\theta)^{\eta+2}}{(\eta+1)(\tau_2-\tau_1)}\left(\int\limits_0^1 (1-\rho^{\eta+1})d\rho\right)\left|\zeta''\left(\frac{\int_0^1 (1-\rho^{\eta+1})(\rho\tau_2+(1-\rho)\theta)d\rho}{\int_0^1 (1-\rho^{\eta+1})d\rho}\right)\right| \\
&= \frac{(\theta-\tau_1)^{\eta+2}\left|\zeta''\left(\frac{(\eta+2)\tau_1+(\eta+4)\theta}{2(\eta+3)}\right)\right| + (\tau_2-\theta)^{\eta+2}\left|\zeta''\left(\frac{(\eta+2)\tau_2+(\eta+4)\theta}{2(\eta+3)}\right)\right|}{(\eta+2)(\tau_2-\tau_1)}.
\end{aligned}
$$

□

Corollary 2.4. *Under the hypothesis of Theorem 2.4, we have the following:*

$$
\begin{aligned}
\left|L_\zeta\left(\frac{\tau_1+\tau_2}{2},\eta,\tau_1,\tau_2\right)\right| &\leq \left(\frac{\tau_2-\tau_1}{2}\right)^{\eta+1}\frac{1}{2(\eta+2)}\left[\left|\zeta''\left(\frac{(3\eta+8)\tau_1+(\eta+4)\tau_2}{4(\eta+3)}\right)\right|\right. \\
&\quad \left. + \left|\zeta''\left(\frac{(3\eta+8)\tau_2+(\eta+4)\tau_1}{4(\eta+3)}\right)\right|\right] \\
&\leq \left(\frac{\tau_2-\tau_1}{2}\right)^{\eta+1}\frac{1}{\eta+2}\left|\zeta''\left(\frac{\tau_1+\tau_2}{2}\right)\right|. \qquad (2.11)
\end{aligned}
$$

Proof. By choosing $\theta=\frac{\tau_1+\tau_2}{2}$ in the inequality (2.10), we get the first bound in (2.11). The second bound in (2.11) is obtained by using the concavity of $|\zeta''|$. □

If we use the property of concavity of $|\zeta''|^s$ in another way, we get an another general result, which is given below.

Theorem 2.5. *Let all the requisites of Lemma 1.1 hold. Additionally, if $|\zeta''|^s$ is concave function on $[\tau_1,\tau_2]$ for $s>1$ such that $r^{-1}+s^{-1}=1$. Then we have the following inequality:*

$$
|L_\zeta(\theta,\eta,\tau_1,\tau_2)| \leq M^{\frac{1}{r}}\frac{(\theta-\tau_1)^{\eta+2}\left|\zeta''\left(\frac{\theta+\tau_1}{2}\right)\right| + (\tau_2-\theta)^{\eta+2}\left|\zeta''\left(\frac{\theta+\tau_2}{2}\right)\right|}{(\eta+1)(\tau_2-\tau_1)} \qquad (2.12)
$$

where $\quad M=\frac{\Gamma(1+r)\Gamma(\frac{1}{\eta+1})}{(\eta+1)\Gamma(1+r+\frac{1}{\eta+1})}$.

Proof. Applying Lemma 1.1, triangle and Hölder inequalities, we have

$$
\begin{aligned}
|L_\zeta(\theta,\eta,\tau_1,\tau_2)| &\leq \frac{(\theta-\tau_1)^{\eta+2}}{(\eta+1)(\tau_2-\tau_1)}\int_0^1 (1-\rho^{\eta+1})|\zeta''(\rho\tau_1+(1-\rho)\theta)|d\rho \\
&+\frac{(\tau_2-\theta)^{\eta+2}}{(\eta+1)(\tau_2-\tau_1)}\int_0^1 (1-\rho^{\eta+1})|\zeta''(\rho\tau_2+(1-\rho)\theta)|d\rho \\
&\leq \frac{(\theta-\tau_1)^{\eta+2}}{(\eta+1)(\tau_2-\tau_1)}\left(\int_0^1 \left(1-\rho^{\eta+1}\right)^r d\rho\right)^{\frac{1}{r}}\left(\int_0^1 |\zeta''(\rho\tau_1+(1-\rho)\theta)|^s d\rho\right)^{\frac{1}{s}} \\
&+\frac{(\tau_2-\theta)^{\eta+2}}{(\eta+1)(\tau_2-\tau_1)}\left(\int_0^1 \left(1-\rho^{\eta+1}\right)^r d\rho\right)^{\frac{1}{r}}\left(\int_0^1 |\zeta''(\rho\tau_2+(1-\rho)\theta)|^s d\rho\right)^{\frac{1}{s}}
\end{aligned}
$$

Since, $|\zeta''|^s$ is concave on $[\tau_1,\tau_2]$, we can use the Jensen's inequality to get:

$$
\begin{aligned}
\int_0^1 |\ \zeta''(\rho\tau_1+(1-\rho)\theta)\ |^s\ d\rho &\leq \left|\zeta''\left(\int_0^1 (\rho\tau_1+(1-\rho)\theta)d\rho\right)\right|^s \\
&= \left|\zeta''\left(\frac{\theta+\tau_1}{2}\right)\right|^s,
\end{aligned}
$$

Similarly,

$$
\int_0^1 |\ \zeta''(\rho\tau_2+(1-\rho)\theta)\ |^s\ d\rho \leq \left|\zeta''\left(\frac{\theta+\tau_2}{2}\right)\right|^s.
$$

Also,

$$
\int_0^1 \left(1-\rho^{\eta+1}\right)^r d\rho \;=\; \frac{1}{\eta+1}\int_0^1 (1-u)^r u^{\frac{1}{\eta+1}-1}du = \frac{\Gamma(1+r)\Gamma(\frac{1}{\eta+1})}{(\eta+1)\Gamma(1+r+\frac{1}{\eta+1})} = M.
$$

putting the last three inequalities in the above inequality, we get the required inequality in (2.12). ☐

Corollary 2.5. *Let all the requisites of Theorem* 2.5 *hold. Then we have:*

$$
\begin{aligned}
\left|L_\zeta\left(\frac{\tau_1+\tau_2}{2},\eta,\tau_1,\tau_2\right)\right| &\leq \frac{M^{\frac{1}{r}}(\tau_2-\tau_1)^{\eta+1}}{2^{\eta+2}(\eta+1)}\left[\left|\zeta''\left(\frac{3\tau_1+\tau_2}{4}\right)\right|+\left|\zeta''\left(\frac{\tau_1+3\tau_2}{4}\right)\right|\right] \\
&\leq \frac{M^{\frac{1}{r}}}{\eta+1}\left(\frac{\tau_2-\tau_1}{2}\right)^{\eta+1}\left|\zeta''\left(\frac{\tau_1+\tau_2}{2}\right)\right|. \qquad (2.13)
\end{aligned}
$$

where

$$
M=\frac{\Gamma(1+r)\Gamma(\frac{1}{\eta+1})}{(\eta+1)\Gamma(1+r+\frac{1}{\eta+1})}.
$$

Proof. By choosing $\theta=\frac{\tau_1+\tau_2}{2}$ in the inequality (2.12), we get the first bound in (2.13). The second bound in (2.13) is obtained using concavity of $|\zeta''|$. ☐

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