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REVERSES OF CALLEBAUT DISCRETE INEQUALITY VIA SOME RESULTS DUE TO ZHUANG

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Abstract. In this paper, by the use of Zhuang's inequalities, we establish some reverse inequalities for the celebrated refinement of the Cauchy-Bunyakovsky-Schwarz inequality that was obtained by Callebaut in 1965. A numerical comparison is also provided.

1. Introduction

The following inequality
$$x^{1-\nu}y^{\nu} \leq (1-\nu)\,x+\nu y \tag{1.1}$$
is well known in literature as either *weighted Arithmetic mean-Geometric mean inequality* or as *Young's inequality.*

In 1991, Y.-D. Zhuang [14] established the following inequality for $0<m\leq x\leq M$, $0<k\leq y\leq K$, and $\nu\in[0,1]$
$$\nu x+(1-\nu)\,y\leq \max\left\{\frac{\nu M+(1-\nu)\,k}{M^{\nu}k^{1-\nu}},\frac{\nu m+(1-\nu)\,K}{m^{\nu}K^{1-\nu}}\right\}x^{\nu}y^{1-\nu} \tag{1.2}$$
or
$$x+y\leq \max\left\{\frac{M+k}{M^{\nu}k^{1-\nu}},\frac{m+K}{m^{\nu}K^{1-\nu}}\right\}x^{\nu}y^{1-\nu}. \tag{1.3}$$
The sign of equality in (1.2) and (1.3) holds if and only if either $(x,y)=(m,K)$ or $(x,y)=(M,k)$.

Moreover, if $m\geq K$, then
$$\frac{\nu m+(1-\nu)\,K}{m^{\nu}K^{1-\nu}}x^{\nu}y^{1-\nu}\leq \nu x+(1-\nu)\,y\leq \frac{\nu M+(1-\nu)\,k}{M^{\nu}k^{1-\nu}}x^{\nu}y^{1-\nu}. \tag{1.4}$$

The sign of equality on the right-hand side of (1.4) holds if and only if $(x,y)=(M,k)$ and the sign of equality on the left-hand side of (1.4) holds if and only if $(x,y)=(m,K)$. The sign of inequality in (1.4) is reversed if $k\geq M$.

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Now, if we take $y = 1$, then we have from the above inequalities for $x \in [m, M] \subset (0, \infty)$ and $\nu \in [0, 1]$ that

$$\nu x + 1 - \nu \leq \max\left\{\frac{\nu M + 1 - \nu}{M^\nu}, \frac{\nu m + 1 - \nu}{m^\nu}\right\} x^\nu \tag{1.5}$$

and

$$x + 1 \leq \max\left\{\frac{M+1}{M^\nu}, \frac{m+1}{m^\nu}\right\} x^\nu. \tag{1.6}$$

If $m \geq 1$, then we have

$$\frac{\nu m + 1 - \nu}{m^\nu} x^\nu \leq \nu x + 1 - \nu \leq \frac{\nu M + 1 - \nu}{M^\nu} x^\nu \tag{1.7}$$

for $x \in [m, M]$ and $\nu \in (0, 1)$.

If $M \leq 1$, then we have

$$\frac{\nu M + 1 - \nu}{M^\nu} x^\nu \leq \nu x + 1 - \nu \leq \frac{\nu m + 1 - \nu}{m^\nu} x^\nu, \tag{1.8}$$

for $x \in [m, M]$ and $\nu \in (0, 1)$.

The inequalities (1.5), (1.7) and (1.8) can be put together as

$$\begin{aligned} \begin{cases} \frac{\nu M+1-\nu}{M^\nu} \text{ if } M < 1, \\ 1 \text{ if } m \leq 1 \leq M, \\ \frac{\nu m+1-\nu}{m^\nu} \text{ if } 1 < m. \end{cases} &\leq \frac{\nu x + 1 - \nu}{x^\nu} \\ &\leq \begin{cases} \frac{\nu m+1-\nu}{m^\nu} \text{ if } M < 1, \\ \max\left\{\frac{\nu M+1-\nu}{M^\nu}, \frac{\nu m+1-\nu}{m^\nu}\right\} \text{ if } m \leq 1 \leq M, \\ \frac{\nu M+1-\nu}{M^\nu} \text{ if } 1 < m, \end{cases} \end{aligned} \tag{1.9}$$

for $x \in [m, M] \subset (0, \infty)$ and $\nu \in [0, 1]$.

The inequality

$$1 \leq \frac{\nu x + 1 - \nu}{x^\nu}$$

is the AG-inequality for $y = 1$,

We notice that the inequality (1.9) has been also obtained in [5] by a direct approach in studying the margins of the function $g(x) := \frac{\nu x+1-\nu}{x^\nu}$ with $x \in [m, M] \subset (0, \infty)$ and $\nu \in [0, 1]$.

For other similar results, see [1] and [3]-[13].

The following refinement of the Cauchy-Bunyakovsky-Schwarz inequality was obtained by Callebaut [2] in 1965:

$$\left(\sum_{i=1}^n p_i a_i b_i\right)^2 \leq \sum_{i=1}^n p_i a_i^{2(1-\nu)} b_i^{2\nu} \sum_{i=1}^n p_i a_i^{2\nu} b_i^{2(1-\nu)} \leq \sum_{i=1}^n p_i a_i^2 \sum_{i=1}^n p_i b_i^2. \tag{1.10}$$

In this paper, by the use of Zhuang's inequalities (1.2) and (1.3) we establish some upper bounds for the quotient

$$\frac{\sum_{i\in I} p_i b_i^2 \sum_{i\in I} p_i a_i^2}{\sum_{i\in I} p_i a_i^{2(1-\nu)} b_i^{2\nu} \sum_{i\in I} p_i a_i^{2\nu} b_i^{2(1-\nu)}}$$

under suitable conditions for the sequences a_k, $b_k > 0$ and $p_k \geq 0$, $k \in \mathbb{N}$.

These results can be applied for operator inequalities as in [1], [5]-[7] and [9].

2. Discrete Inequalities

We start with the following result:

Theorem 2.1. *Let* $a_k,\ b_k > 0,\ k \in \mathbb{N}$ *and* $I,\ J$ *be finite sets of indices such that*

$$m \le \frac{b_i}{a_i} \le M \text{ and } k \le \frac{b_j}{a_j} \le K \tag{2.1}$$

for some constants $0 < m < M,\ 0 < k < K$, *for any* $i \in I$ *and* $j \in J$. *If* $p_i \ge 0$ *for* $i \in I$, $q_j \ge 0$ *for* $j \in J$ *and* $\nu \in [0,1]$, *then we have the inequality*

$$\begin{aligned} &\nu \sum_{i\in I} p_i b_i^2 \sum_{j\in J} q_j a_j^2 + (1-\nu) \sum_{i\in I} p_i a_i^2 \sum_{j\in J} q_j b_j^2 \\ &\le \max\left\{\frac{\nu M^2 + (1-\nu) k^2}{M^{2\nu} k^{2(1-\nu)}}, \frac{\nu m^2 + (1-\nu) K^2}{m^{2\nu} K^{2(1-\nu)}}\right\} \\ &\times \sum_{i\in I} p_i a_i^{2(1-\nu)} b_i^{2\nu} \sum_{j\in J} q_j a_j^{2\nu} b_j^{2(1-\nu)} \end{aligned} \tag{2.2}$$

and the inequality

$$\begin{aligned} &\sum_{i\in I} p_i b_i^2 \sum_{j\in J} q_j a_j^2 + \sum_{i\in I} p_i a_i^2 \sum_{j\in J} q_j b_j^2 \\ &\le \max\left\{\frac{M^2 + k^2}{M^{2\nu} k^{2(1-\nu)}}, \frac{m^2 + K^2}{m^{2\nu} K^{2(1-\nu)}}\right\} \sum_{i\in I} p_i a_i^{2(1-\nu)} b_i^{2\nu} \sum_{j\in J} q_j a_j^{2\nu} b_j^{2(1-\nu)}. \end{aligned} \tag{2.3}$$

Proof. If we write the inequality (1.2) for $x = \left(\frac{b_i}{a_i}\right)^2$ and $y = \left(\frac{b_j}{a_j}\right)^2$, then we get

$$\begin{aligned} &\nu \left(\frac{b_i}{a_i}\right)^2 + (1-\nu) \left(\frac{b_j}{a_j}\right)^2 \\ &\le \max\left\{\frac{\nu M^2 + (1-\nu) k^2}{M^{2\nu} k^{2(1-\nu)}}, \frac{\nu m^2 + (1-\nu) K^2}{m^{2\nu} K^{2(1-\nu)}}\right\} \left(\frac{b_i}{a_i}\right)^{2\nu} \left(\frac{b_j}{a_j}\right)^{2(1-\nu)} \end{aligned} \tag{2.4}$$

for any $i \in I$ and $j \in J$.

By multiplying (2.4) with $a_i^2 a_j^2 \ge 0$ we get

$$\begin{aligned} &\nu b_i^2 a_j^2 + (1-\nu) a_i^2 b_j^2 \\ &\le \max\left\{\frac{\nu M^2 + (1-\nu) k^2}{M^{2\nu} k^{2(1-\nu)}}, \frac{\nu m^2 + (1-\nu) K^2}{m^{2\nu} K^{2(1-\nu)}}\right\} a_i^{2(1-\nu)} b_i^{2\nu} a_j^{2\nu} b_j^{2(1-\nu)} \end{aligned} \tag{2.5}$$

for any $i \in I$ and $j \in J$.

Multiply the inequality (2.5) by $q_j \ge 0$ and sum over $j \in J$ to get

$$\begin{aligned} &\nu b_i^2 \sum_{j\in J} q_j a_j^2 + (1-\nu) a_i^2 \sum_{j\in J} q_j b_j^2 \\ &\le \max\left\{\frac{\nu M^2 + (1-\nu) k^2}{M^{2\nu} k^{2(1-\nu)}}, \frac{\nu m^2 + (1-\nu) K^2}{m^{2\nu} K^{2(1-\nu)}}\right\} a_i^{2(1-\nu)} b_i^{2\nu} \sum_{j\in J} q_j a_j^{2\nu} b_j^{2(1-\nu)} \end{aligned} \tag{2.6}$$

for any $i \in I$.

If we multiply (2.6) by $p_i \geq 0$ and sum over $i \in I$, we get the desired inequality (2.2).

By the inequality (1.3) for $x = \left(\frac{b_i}{a_i}\right)^2$ and $y = \left(\frac{b_j}{a_j}\right)^2$ we have

$$\left(\frac{b_i}{a_i}\right)^2 + \left(\frac{b_j}{a_j}\right)^2 \leq \max\left\{\frac{M^2+k^2}{M^{2\nu}k^{2(1-\nu)}}, \frac{m^2+K^2}{m^{2\nu}K^{2(1-\nu)}}\right\}\left(\frac{b_i}{a_i}\right)^{2\nu}\left(\frac{b_j}{a_j}\right)^{2(1-\nu)} \tag{2.7}$$

for any $i \in I$ and $j \in J$. On making use of a similar argument as above, we deduce (2.3). □

Corollary 2.1. *Let $a_k,\ b_k > 0,\ k \in \mathbb{N}$ and I be a finite set of indices such that*

$$m \leq \frac{b_i}{a_i} \leq M \tag{2.8}$$

for some constants $0 < m < M$ and any $i \in I$. If $p_i \geq 0$ for $i \in I$ and $\nu \in [0,1]$, then we have the inequality

$$\begin{aligned}&\frac{\sum_{i\in I} p_i b_i^2 \sum_{i\in I} p_i a_i^2}{\sum_{i\in I} p_i a_i^{2(1-\nu)} b_i^{2\nu} \sum_{i\in I} p_i a_i^{2\nu} b_i^{2(1-\nu)}} \\ &\leq \max\left\{\frac{\nu M^2 + (1-\nu)\, m^2}{M^{2\nu}m^{2(1-\nu)}}, \frac{\nu m^2 + (1-\nu)\, M^2}{m^{2\nu}M^{2(1-\nu)}}\right\}\end{aligned} \tag{2.9}$$

and the inequality

$$\begin{aligned}&\frac{\sum_{i\in I} p_i b_i^2 \sum_{i\in I} p_i a_i^2}{\sum_{i\in I} p_i a_i^{2(1-\nu)} b_i^{2\nu} \sum_{i\in I} p_i a_i^{2\nu} b_i^{2(1-\nu)}} \\ &\leq \frac{M^2+m^2}{2}\max\left\{\frac{1}{M^{2\nu}m^{2(1-\nu)}}, \frac{1}{m^{2\nu}M^{2(1-\nu)}}\right\}.\end{aligned} \tag{2.10}$$

The inequalities (2.9) and (2.10) therefore provide multiplicative reverses of the second Callebaut inequality (1.10).

The following result also holds:

Theorem 2.2. *Let $a_k,\ b_k > 0,\ k \in \mathbb{N}$ and I be a finite set of indices such that*

$$a \leq a_i \leq A \text{ and } b \leq b_i \leq B \tag{2.11}$$

for some constants $0 < a < A$, $0 < b < B$ and any $i \in I$. If $w_i \geq 0$ for $i \in I$ with $\sum_{i\in I} w_i = 1$ and $\nu \in [0,1]$, then we have the inequality

$$\begin{aligned}&\frac{\left(\sum_{i\in I} w_i a_i^2\right)^{\nu}\left(\sum_{i\in I} w_i b_i^2\right)^{1-\nu}}{\sum_{j\in I} w_j a_j^{2\nu} b_j^{2(1-\nu)}} \\ &\leq \max\left\{\frac{\nu A^2B^2 + (1-\nu)\, a^2b^2}{A^{2\nu}a^{2(1-\nu)}B^{2\nu}b^{2(1-\nu)}}, \frac{\nu a^2b^2 + (1-\nu)\, A^2B^2}{A^{2(1-\nu)}a^{2\nu}B^{2(1-\nu)}b^{2\nu}}\right\}\end{aligned} \tag{2.12}$$

and the inequality

$$\frac{\left(\sum_{i\in I} w_i a_i^2\right)^\nu \left(\sum_{i\in I} w_i b_i^2\right)^{1-\nu}}{\sum_{j\in I} w_j a_j^{2\nu} b_j^{2(1-\nu)}} \tag{2.13}$$
$$\leq \frac{A^2B^2 + a^2b^2}{2}$$
$$\times \max\left\{\frac{1}{A^{2\nu}a^{2(1-\nu)}B^{2\nu}b^{2(1-\nu)}}, \frac{1}{A^{2(1-\nu)}a^{2\nu}B^{2(1-\nu)}b^{2\nu}}\right\}.$$

Proof. Let $x = \frac{a_j^2}{\sum_{i\in I} w_i a_i^2}$ and $y = \frac{b_j^2}{\sum_{i\in I} w_i b_i^2}$ for $j \in I$, then we get

$$\frac{a^2}{A^2} \leq x \leq \frac{A^2}{a^2},\ j \in I$$

and

$$\frac{b^2}{B^2} \leq y \leq \frac{B^2}{b^2},\ j \in I.$$

If we write the inequality (1.2) for x and y as above, then we get

$$\nu \frac{a_j^2}{\sum_{i\in I} w_i a_i^2} + (1-\nu)\frac{b_j^2}{\sum_{i\in I} w_i b_i^2} \tag{2.14}$$
$$\leq \max\left\{\frac{\nu\frac{A^2}{a^2} + (1-\nu)\frac{b^2}{B^2}}{\left(\frac{A^2}{a^2}\right)^\nu\left(\frac{b^2}{B^2}\right)^{1-\nu}}, \frac{\nu\frac{a^2}{A^2} + (1-\nu)\frac{B^2}{b^2}}{\left(\frac{a^2}{A^2}\right)^\nu\left(\frac{B^2}{b^2}\right)^{1-\nu}}\right\}$$
$$\times \frac{a_j^{2\nu}}{\left(\sum_{i\in I} w_i a_i^2\right)^\nu}\frac{b_j^{2(1-\nu)}}{\left(\sum_{i\in I} w_i b_i^2\right)^{1-\nu}}$$

for any $j \in I$.

Since

$$\frac{\nu\frac{A^2}{a^2} + (1-\nu)\frac{b^2}{B^2}}{\left(\frac{A^2}{a^2}\right)^\nu\left(\frac{b^2}{B^2}\right)^{1-\nu}} = \frac{\nu A^2B^2 + (1-\nu)a^2b^2}{A^{2\nu}a^{2(1-\nu)}B^{2\nu}b^{2(1-\nu)}}$$

and

$$\frac{\nu\frac{a^2}{A^2} + (1-\nu)\frac{B^2}{b^2}}{\left(\frac{a^2}{A^2}\right)^\nu\left(\frac{B^2}{b^2}\right)^{1-\nu}} = \frac{\nu a^2b^2 + (1-\nu)A^2B^2}{A^{2(1-\nu)}a^{2\nu}B^{2(1-\nu)}b^{2\nu}},$$

then by (2.14) we have

$$\nu \frac{a_j^2}{\sum_{i\in I} w_i a_i^2} + (1-\nu)\frac{b_j^2}{\sum_{i\in I} w_i b_i^2} \tag{2.15}$$
$$\leq \max\left\{\frac{\nu A^2B^2 + (1-\nu)a^2b^2}{A^{2\nu}a^{2(1-\nu)}B^{2\nu}b^{2(1-\nu)}}, \frac{\nu a^2b^2 + (1-\nu)A^2B^2}{A^{2(1-\nu)}a^{2\nu}B^{2(1-\nu)}b^{2\nu}}\right\}$$
$$\times \frac{a_j^{2\nu}}{\left(\sum_{i\in I} w_i a_i^2\right)^\nu}\frac{b_j^{2(1-\nu)}}{\left(\sum_{i\in I} w_i b_i^2\right)^{1-\nu}}$$

for any $j \in I$.

If we multiply (2.15) by w_j and sum, then we get

$$\begin{aligned}&\nu\frac{\sum_{j\in I} w_j a_j^2}{\sum_{i\in I} w_i a_i^2}+(1-\nu)\frac{\sum_{j\in I} w_j b_j^2}{\sum_{i\in I} w_i b_i^2}\\ &\leq \max\left\{\frac{\nu A^2B^2+(1-\nu)a^2b^2}{A^{2\nu}a^{2(1-\nu)}B^{2\nu}b^{2(1-\nu)}},\frac{\nu a^2b^2+(1-\nu)A^2B^2}{A^{2(1-\nu)}a^{2\nu}B^{2(1-\nu)}b^{2\nu}}\right\}\\ &\times\frac{\sum_{j\in I} w_j a_j^{2\nu} b_j^{2(1-\nu)}}{\left(\sum_{i\in I} w_i a_i^2\right)^{\nu}\left(\sum_{i\in I} w_i b_i^2\right)^{1-\nu}}\end{aligned}$$

that is equivalent to (2.12).

By the inequality (1.3) we also have

$$\begin{aligned}&\frac{a_j^2}{\sum_{i\in I} w_i a_i^2}+\frac{b_j^2}{\sum_{i\in I} w_i b_i^2}\\ &\leq \max\left\{\frac{\frac{A^2}{a^2}+\frac{b^2}{B^2}}{\left(\frac{A^2}{a^2}\right)^{\nu}\left(\frac{b^2}{B^2}\right)^{1-\nu}},\frac{\frac{a^2}{A^2}+\frac{B^2}{b^2}}{\left(\frac{a^2}{A^2}\right)^{\nu}\left(\frac{B^2}{b^2}\right)^{1-\nu}}\right\}\\ &\times\frac{a_j^{2\nu}}{\left(\sum_{i\in I} w_i a_i^2\right)^{\nu}}\frac{b_j^{2(1-\nu)}}{\left(\sum_{i\in I} w_i b_i^2\right)^{1-\nu}}\end{aligned}\tag{2.16}$$

for any $j\in I$ and since

$$\begin{aligned}&\max\left\{\frac{\frac{A^2}{a^2}+\frac{b^2}{B^2}}{\left(\frac{A^2}{a^2}\right)^{\nu}\left(\frac{b^2}{B^2}\right)^{1-\nu}},\frac{\frac{a^2}{A^2}+\frac{B^2}{b^2}}{\left(\frac{a^2}{A^2}\right)^{\nu}\left(\frac{B^2}{b^2}\right)^{1-\nu}}\right\}\\ &=\left(A^2B^2+a^2b^2\right)\\ &\times\max\left\{\frac{1}{A^{2\nu}a^{2(1-\nu)}B^{2\nu}b^{2(1-\nu)}},\frac{1}{A^{2(1-\nu)}a^{2\nu}B^{2(1-\nu)}b^{2\nu}}\right\},\end{aligned}$$

then by (2.16) we get

$$\begin{aligned}&\frac{a_j^2}{\sum_{i\in I} w_i a_i^2}+\frac{b_j^2}{\sum_{i\in I} w_i b_i^2}\\ &\leq\left(A^2B^2+a^2b^2\right)\\ &\times\max\left\{\frac{1}{A^{2\nu}a^{2(1-\nu)}B^{2\nu}b^{2(1-\nu)}},\frac{1}{A^{2(1-\nu)}a^{2\nu}B^{2(1-\nu)}b^{2\nu}}\right\}\\ &\times\frac{a_j^{2\nu}}{\left(\sum_{i\in I} w_i a_i^2\right)^{\nu}}\frac{b_j^{2(1-\nu)}}{\left(\sum_{i\in I} w_i b_i^2\right)^{1-\nu}}\end{aligned}\tag{2.17}$$

for any $j\in I$.

If we multiply (2.17) by w_j and sum, then we get the desired result (2.13). □

Remark 2.1. With the assumptions of Theorem 2.2 we have the Callebaut reverse inequalities

$$\frac{\sum_{i\in I} w_i a_i^2 \sum_{i\in I} w_i b_i^2}{\sum_{j\in I} w_j a_j^{2\nu} b_j^{2(1-\nu)} \sum_{j\in I} w_j a_j^{2(1-\nu)} b_j^{2\nu}} \le \max\left\{\frac{\left(\nu A^2B^2 + (1-\nu)\, a^2b^2\right)^2}{A^{4\nu}a^{4(1-\nu)}B^{4\nu}b^{4(1-\nu)}}, \frac{\left(\nu a^2b^2 + (1-\nu)\, A^2B^2\right)^2}{A^{4(1-\nu)}a^{4\nu}B^{4(1-\nu)}b^{4\nu}}\right\} \tag{2.18}$$

and

$$\frac{\sum_{i\in I} w_i a_i^2 \sum_{i\in I} w_i b_i^2}{\sum_{j\in I} w_j a_j^{2\nu} b_j^{2(1-\nu)} \sum_{j\in I} w_j a_j^{2(1-\nu)} b_j^{2\nu}} \le \left(\frac{A^2B^2 + a^2b^2}{2}\right)^2 \times \max\left\{\frac{1}{A^{4\nu}a^{4(1-\nu)}B^{4\nu}b^{4(1-\nu)}}, \frac{1}{A^{4(1-\nu)}a^{4\nu}B^{4(1-\nu)}b^{4\nu}}\right\}. \tag{2.19}$$

Indeed, by the inequality (2.12) for $1-\nu$ instead of ν we have

$$\frac{\left(\sum_{i\in I} w_i a_i^2\right)^{1-\nu} \left(\sum_{i\in I} w_i b_i^2\right)^{\nu}}{\sum_{j\in I} w_j a_j^{2(1-\nu)} b_j^{2\nu}} \le \max\left\{\frac{(1-\nu)\, A^2B^2 + \nu a^2b^2}{A^{2(1-\nu)}a^{2\nu}B^{2(1-\nu)}b^{2\nu}}, \frac{(1-\nu)\, a^2b^2 + \nu A^2B^2}{A^{2\nu}a^{2(1-\nu)}B^{2\nu}b^{2(1-\nu)}}\right\}. \tag{2.20}$$

If we multiply (2.12) with (2.20) we obtain (2.18).

The inequality (2.19) follows in a similar way by (2.13) and the details are omitted.

The inequalities from (2.19) and (2.20) can be however improved as follows:

Theorem 2.3. *Let $a_k, b_k > 0$, $k \in \mathbb{N}$ and I a finite set of indices such that the inequality (2.11) is valid for some constants $0 < a < A$, $0 < b < B$ for any $i \in I$. If $w_i \ge 0$ for $i \in I$ with $\sum_{i\in I} w_i = 1$ and $\nu \in [0,1]$, then we have the inequalities*

$$\frac{\sum_{i\in I} w_i a_i^2 \sum_{i\in I} w_i b_i^2}{\sum_{j\in I} w_j a_j^{2\nu} b_j^{2(1-\nu)} \sum_{j\in I} w_j a_j^{2(1-\nu)} b_j^{2\nu}} \le \max\left\{\frac{\nu A^2B^2 + (1-\nu)\, a^2b^2}{A^{2\nu}B^{2\nu}a^{2(1-\nu)}b^{2(1-\nu)}}, \frac{\nu a^2b^2 + (1-\nu)\, A^2B^2}{a^{2\nu}b^{2\nu}A^{2(1-\nu)}B^{2(1-\nu)}}\right\} \tag{2.21}$$

and

$$\frac{\sum_{i\in I} w_i a_i^2 \sum_{i\in I} w_i b_i^2}{\sum_{j\in I} w_j a_j^{2\nu} b_j^{2(1-\nu)} \sum_{j\in I} w_j a_j^{2(1-\nu)} b_j^{2\nu}} \le \frac{A^2B^2 + a^2b^2}{2} \times \max\left\{\frac{1}{A^{2\nu}B^{2\nu}a^{2(1-\nu)}b^{2(1-\nu)}}, \frac{1}{a^{2\nu}b^{2\nu}A^{2(1-\nu)}B^{2(1-\nu)}}\right\}. \tag{2.22}$$

Proof. Let $x = a_i^2 b_j^2$ and $y = a_j^2 b_i^2$ for $i,\, j \in I$. Then by the condition (2.11) we have

$$a^2b^2 \le x \le A^2B^2 \text{ and } a^2b^2 \le y \le A^2B^2.$$

By the inequalities (1.2) and (1.3) we have

$$\begin{aligned} &\nu a_i^2 b_j^2 + (1-\nu)\, a_j^2 b_i^2 \\ &\le \max\left\{ \frac{\nu A^2B^2 + (1-\nu)\, a^2b^2}{(A^2B^2)^{\nu}\, (a^2b^2)^{1-\nu}}, \frac{\nu a^2b^2 + (1-\nu)\, A^2B^2}{(a^2b^2)^{\nu}\, (A^2B^2)^{1-\nu}} \right\} \\ &\times \left(a_i^2 b_j^2\right)^{\nu} \left(a_j^2 b_i^2\right)^{1-\nu} \\ &= \max\left\{ \frac{\nu A^2B^2 + (1-\nu)\, a^2b^2}{A^{2\nu}B^{2\nu}a^{2(1-\nu)}b^{2(1-\nu)}}, \frac{\nu a^2b^2 + (1-\nu)\, A^2B^2}{a^{2\nu}b^{2\nu}A^{2(1-\nu)}B^{2(1-\nu)}} \right\} \\ &\times a_i^{2\nu} b_i^{2(1-\nu)} a_j^{2(1-\nu)} b_j^{2\nu} \end{aligned} \tag{2.23}$$

and

$$\begin{aligned} a_i^2 b_j^2 + a_j^2 b_i^2 \le &\left(A^2B^2 + a^2b^2\right) \\ &\times \max\left\{ \frac{1}{A^{2\nu}B^{2\nu}a^{2(1-\nu)}b^{2(1-\nu)}}, \frac{1}{a^{2\nu}b^{2\nu}A^{2(1-\nu)}B^{2(1-\nu)}} \right\} \\ &\times a_i^{2\nu} b_i^{2(1-\nu)} a_j^{2(1-\nu)} b_j^{2\nu}, \end{aligned} \tag{2.24}$$

for $i,\, j \in I$.

If we multiply (2.23) and (2.24) by $w_i w_j$ and sum over $i,\, j \in I$ we get the desired inequalities (2.21) and (2.22). □

3. A Numerical Comparison

We consider the *Kantorovich's ratio* defined by

$$K(h) := \frac{(h+1)^2}{4h},\ h > 0. \tag{3.1}$$

The function K is decreasing on $(0,1)$ and increasing on $[1,\infty)$, $K(h) \ge 1$ for any $h > 0$ and $K(h) = K\left(\frac{1}{h}\right)$ for any $h > 0$.

The following multiplicative reverse of Young inequality in terms of Kantorovich's ratio holds

$$(1-\nu)\, a + \nu b \le K^R\left(\frac{a}{b}\right) a^{1-\nu} b^{\nu}, \tag{3.2}$$

where $a, b > 0$, $\nu \in [0,1]$ and $R = \max\{1-\nu, \nu\}$.

This inequality was obtained by Liao et al. [11].

In [8] the first author obtained the following reverse of Callebaut inequality

$$\frac{\sum_{i\in I} p_i b_i^2 \sum_{i\in I} p_i a_i^2}{\sum_{i\in I} p_i a_i^{2(1-\nu)} b_i^{2\nu} \sum_{i\in I} p_i a_i^{2\nu} b_i^{2(1-\nu)}} \le K^{\max\{v,1-v\}}\left(\left(\frac{M}{m}\right)^2\right) \tag{3.3}$$

where $a_k,\, b_k > 0$, $k \in \mathbb{N}$ and I a finite set of indices such that the condition (2.8) is valid for some constants $0 < m < M$ and any $i \in I$, $w_i \ge 0$ for $i \in I$ with $\sum_{i\in I} w_i = 1$ and $\nu \in [0,1]$.

From (2.9), (2.10) and (3.3) we have the following upper bounds for the quotient

$$\frac{\sum_{i\in I} p_i b_i^2 \sum_{i\in I} p_i a_i^2}{\sum_{i\in I} p_i a_i^{2(1-\nu)} b_i^{2\nu} \sum_{i\in I} p_i a_i^{2\nu} b_i^{2(1-\nu)}} \tag{3.4}$$
$$\leq B_1(m,M,\nu),\ B_2(m,M,\nu),\ B_3(m,M,\nu)$$

where

$$B_1(m,M,\nu) := \max\left\{\frac{\nu M^2+(1-\nu)m^2}{M^{2\nu}m^{2(1-\nu)}}, \frac{\nu m^2+(1-\nu)M^2}{m^{2\nu}M^{2(1-\nu)}}\right\},$$
$$B_2(m,M,\nu) := \frac{M^2+m^2}{2}\max\left\{\frac{1}{M^{2\nu}m^{2(1-\nu)}}, \frac{1}{m^{2\nu}M^{2(1-\nu)}}\right\},$$

and

$$B_3(m,M,\nu) := K^{\max\{v,1-v\}}\left(\left(\frac{M}{m}\right)^2\right).$$

Here $0 < m \leq M < \infty$ and $v \in [0,1]$.

For $m = 1$, we consider the differences

$$D_1(M,\nu) \ :\ = B_1(1,M,\nu) - B_2(1,M,\nu),$$
$$D_2(M,\nu) \ :\ = B_3(1,M,\nu) - B_1(1,M,\nu),$$
$$D_3(M,\nu) \ :\ = B_3(1,M,\nu) - B_2(1,M,\nu)$$

for $M \geq 1$ and $v \in [0,1]$.

The plots of the differences $D_1(M,\nu)$, $D_2(M,\nu)$ and $D_3(M,\nu)$ in the box $[1,3]\times[0,1]$ are depicted in Figures 1, 2 and 3 below. They show that in (3.4) the bound B_1 is better than B_3 that is better than $B_{2.}$

Problem 1. Is the following inequality

$$B_1(m,M,\nu) \leq B_3(m,M,\nu) \leq B_2(m,M,\nu)$$

valid for any $0 < m \leq M < \infty$ and $v \in [0,1]$?

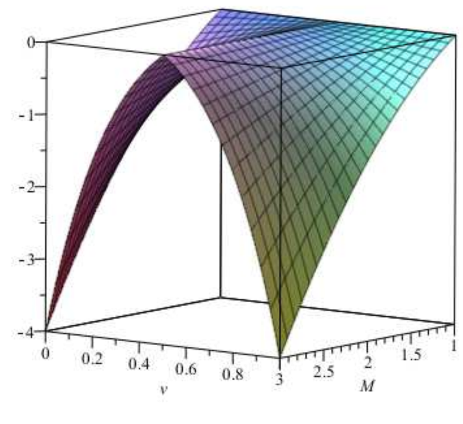

FIGURE 1. Plot of $D_1(M, v)$ in $[1, 3] \times [0, 1]$

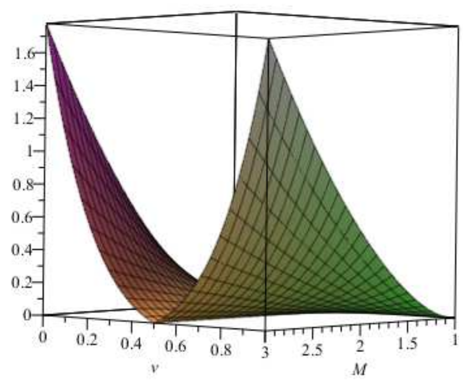

FIGURE 2. Plot of $D_2(M, \nu)$ on $[1, 3] \times [0, 1]$

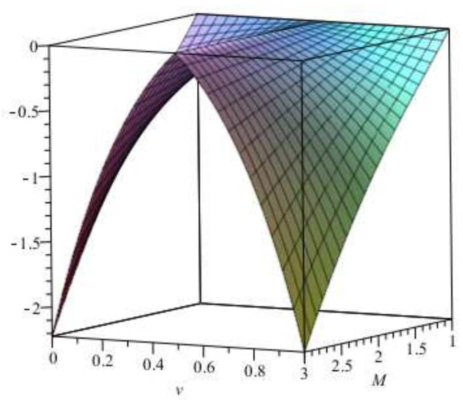

FIGURE 3. Plot of $D_3(M, \nu)$ on $[1, 3] \times [0, 1]$

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