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SOME FUNCTIONAL INEQUALITIES FOR EXTENDED HYPERGEOMETRIC FUNCTION

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ABSTRACT. In this paper, we obtain some functional inequalities for extended hypergeometric function by using classical analysis and inequalities theory.

1. INTRODUCTION

For given complex numbers a, b and c with $c \neq 0, -1, -2, \ldots$, the *Gaussian hypergeometric function* (GHF) is the analytic continuation to the slit place $\mathbb{C} \setminus [1, \infty)$ of the series

$$F(a,b;c;z) = {}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a,n)(b,n)}{(c,n)} \frac{z^n}{n!}, \qquad |z| < 1.$$

Here $(a, 0) = 1$ for $a \neq 0$, and (a, n) is the *shifted factorial function* or the *Appell symbol*

$$(a,n) = a(a+1)(a+2)\cdots(a+n-1)$$

for $n \in \mathbb{Z}_+$, see [1]. The integral representation of the hypergeometric function is given as follows

$$F(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}dt, \tag{1.1}$$
$$\mathrm{Re}(c) > \mathrm{Re}(a) > 0, |\arg(1-z) < \pi|.$$

By using the following integral representation of Euler's beta function

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt, \quad \mathrm{Re}(x), \mathrm{Re}(y) > 0$$

and series expansion of $(1-zt)^{-a}$, GHF can be expressed in terms of beta function as follows

$$F(a,b;c;x) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \sum_{n=0}^{\infty} (a,n) B(b+n, c-b) \frac{x^n}{n!}, \tag{1.2}$$
$$\mathrm{Re}(c) > \mathrm{Re}(b) > 0, \ |x| < 1.$$

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In 1997, Chaudhry et.al [4] introduced the following extended beta function (EBF) $B_p(x, y)$, defined as below

$$B_p(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}e^{-p/(t(1-t))}dt, \tag{1.3}$$

$$\mathrm{Re}(p), \mathrm{Re}(x), \mathrm{Re}(y) > 0.$$

Clearly for $p = 0$, this function coincides with the classical beta function. For the more integral representation of this extended beta function $B_p(x, y)$ and properties, see [4,6].

On the basis of EBF, Chaudhry et.al [5] extended the GHF in 2004. We call it here Extended Gauss hypergeometric function (EGHF), and by using (1.2) and (1.3) defined as below

$$F_p(a, b; c; x) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \sum_{n=0}^{\infty} (a, n) B_p(b+n, c-b) \frac{x^n}{n!} \tag{1.4}$$

$$= \sum_{n=0}^{n} (a, n) \frac{B_p(b+n, c-b)}{B(b, c-b)} \frac{x^n}{n!},$$

$$p \geq 0,\ \mathrm{Re}(c) > \mathrm{Re}(b) > 0,\ |x| < 1.$$

For $p = 0$, EGHF coincides with GHF. By using (1.3), EGHF can written in the integral representation form as follows

$$F_p(a, b; c; x) = \frac{1}{B(b, c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}e^{-p/(t(1-t))} \sum_{n=0}^{\infty} (a, n) \frac{(xt)^n}{n!} dt. \tag{1.5}$$

The above formula can be rewritten in the form as

$$F_p(a, b; c; x) = \frac{1}{B(b, c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-xt)^{-a}e^{-p/(t(1-t))}dt, \tag{1.6}$$

$$p \geq 0,\ \mathrm{Re}(c) > \mathrm{Re}(b) > 0,\ |x| < 1.$$

For the more properties, transformation formulas in terms of other special functions and integral representation of EGHF see [5].

Setting $x = 1$, we get the summation formula for EGHF as follows

$$F_p(a, b; c; 1) = \frac{B_p(b, c-a-b)}{B(b, c-b)}, \quad p \geq 0, \quad \mathrm{Re(c-a-b)} > 0, \tag{1.7}$$

which coincides with the Gauss's summation formula for $p = 0$.

2. Lemmas

Lemma 2.1. [2, Lemma 1] *Consider the power series* $f(x) = \sum_{n\geq 0} a_n x^n$ *and* $g(x) = \sum_{n\geq 0} b_n x^n$, *where* $a_n \in \mathbf{R}$ *and* $b_n > 0$ *for all* $n \in \mathbf{N}\backslash\{0\}$, *and suppose that both converge on* $(-r, r), r > 0$. *If the sequence* $\left\{\frac{a_n}{b_n}\right\}_{n\geq 0}$ *is increasing(decreasing), then the function* $x \mapsto \frac{f(x)}{g(x)}$ *is increasing(decreasing) too on* $(0, r)$.

Lemma 2.2 ([3, Lemma 3, p246]). *Let us consider the function* $f:(a,\infty)\to\mathbf{R}$, *where* $a\geq 0$. *If the function* g, *defined by* $g(x)=\frac{1}{x}(f(x)-1)$, *is increasing on* (a,∞), *then for the function* h, *defined by* $h(x)=f(x^2)$, *we have the following Grünbaum type inequality*

$$1+h(z)\geq h(x)+h(y), \tag{2.1}$$

where $x,y\geq a$ *and* $z^2=x^2+y^2$. *If the function* g *is decreasing, then the inequality (2.1) is reversed.*

3. MAIN RESULTS

Theorem 3.1. *Let* $a,b,c\in\mathbf{R},p\geq 0$ *such that* $c>b>0,a>b>0$ *and consider the function* $H:(0,1)\mapsto(0,\infty)$, *defined by* $H(x)=\frac{F_p(a,b;c;x)}{F_p(a,b;a;x)}$. *Then the function* $H(x)$ *is decreasing and*

$$\frac{F_p(a,b;c;x)}{F_p(a,b;a;x)}\geq\frac{B(b,c-a-b)B(b,a-b)}{B(b,c-b)2\exp(-2p)k_b(2p)} \tag{3.1}$$

holds true for each other $x\in(0,1)$ *where* $k_v(z)$ *is the modified Bessel function.*

Proof. Applying the definition of extended hypergeometric function, we get

$$H(x)=\frac{F_p(a,b;c;x)}{F_p(a,b;a;x)}=\frac{\frac{1}{\mathrm{B}(b,c-b)}\sum\limits_{n=0}^{\infty}(a,n)\mathrm{B_p}(b+n,c-b)\frac{x^n}{n!}}{\frac{1}{\mathrm{B}(b,a-b)}\sum\limits_{n=0}^{\infty}(a,n)\mathrm{B_p}(b+n,a-b)\frac{x^n}{n!}}.$$

So the monotonicity of the function $H(x)$ depends on the monotonicity of the sequence $\{\omega_n\}_{n\geq 0}$, defined by

$$\omega_n=\frac{\mathrm{B}(b,a-b)}{\mathrm{B}(b,c-b)}\frac{\mathrm{B_p}(b+n,c-b)}{\mathrm{B_p}(b+n,a-b)}.$$

Setting $x=b+n+1,x_1=b+n,y=c-b,y_1=a-b$ in Theorem 2.1 of [7], we easily obtain

$$\frac{\mathrm{B_p}(b+n+1,c-b)}{\mathrm{B_p}(b+n+1,a-b)}\leq\frac{\mathrm{B_p}(b+n,c-b)}{\mathrm{B_p}(b+n,a-b)}.$$

So, we have

$$\frac{\omega_{n+1}}{\omega_n}=\frac{\mathrm{B_p}(b+n+1,c-b)}{\mathrm{B_p}(b+n+1,a-b)}\frac{\mathrm{B_p}(b+n,a-b)}{\mathrm{B_p}(b+n,c-b)}\leq 1.$$

in view of Lemma 2.1, the function H is decreasing for all $x\in(0,1)$. Therefore, we have $H(x)\geq H(1)$. Using (1.7) and the formula (8.5)

$$F_p(a,b;a;1)=\frac{2\exp(-2p)}{B(b,a-b)}k_b(2p)$$

in reference [5], we complete the proof. □

Using completely similar method to Theorem 3.1, we easily obtain the following Theorem 3.2.

Theorem 3.2. *Let $a, b, c \in \mathbf{R}, p \geq 0$ such that $c > b > 0$ and consider the function $I : (0,1) \mapsto (0,\infty)$, defined by $I(x) = \frac{F_p(a,b;c;x)}{F_p(a,b;a+b;x)}$. Then the function $I(x)$ is decreasing and*

$$\frac{F_p(a,b;c;x)}{F_p(a,b;a+b;x)} \geq \frac{2^b B(b,c-a-b)B(b,a+b)}{B(b,c-b)\sqrt{\pi}p^{(b-1)/2}\exp(-2p)W_{-b/2,b/2}(2p)} \tag{3.2}$$

holds true for each other $x \in (0,1)$ where $W_{\mu,\kappa}$ is the Whittaker function[4].

Theorem 3.3. *For $c > b > 0$ and $z^2 = x^2 + y^2$, then the following inequality holds:*

$$\frac{\mathrm{B_p}(b,c-b)}{\mathrm{B}(b,c-b)} + F_p(a,b;c;z^2) \geq F_p(a,b;c;x^2) + F_p(a,b;c;y^2). \tag{3.3}$$

Proof. Suppose

$$f(x) = \frac{\mathrm{B}(b,c-b)}{\mathrm{B_p}(b,c-b)} F_p(a,b;c;x).$$

Applying Lemma 2.2, we only need prove that the function $\frac{f(x)-1}{x}$ is strictly increasing on $(0,\infty)$. The differentiation formula

$$\frac{d}{dx}\left(\frac{f(x)-1}{x}\right) = \sum_{n=2}^{\infty}(a,n)\frac{\mathrm{B_p}(b,c-b)}{\mathrm{B}(b,c-b)}\frac{(n-1)x^{n-2}}{n!} > 0$$

implies that the function $\frac{f(x)-1}{x}$ is increasing on $x \in (0,1)$. We complete the proof. □

Theorem 3.4. *For $c > b > 0$ and $x \in (0,1)$ fixed, then the function $p \mapsto F_p(a,b;c;x)$ is strictly completely monotonic on $p \in [0,\infty)$.*

Proof. Since

$$(-1)^n \frac{d^n}{dp^n}\left(exp\left(-\frac{p}{t(1-t)}\right)\right) = \left(\frac{1}{t(1-t)}\right)^n exp\left(-\frac{p}{t(1-t)}\right) > 0,$$

we obtain that the function $e^{-\frac{p}{t(1-t)}}$ is completely monotonic on $p \in (0,\infty)$. This implies that the function $p \mapsto F_p(a,b;c;x)$ is completely monotonic on $p \in (0,\infty)$. □

References

[1] M. Abramowitz, I. Stegun, *Handbook of mathematical functions with formulas, graphs and mathematical tables,* National Bureau of Standards, Dover, New York, 1965.

[2] Á. Baricz, *Landen type inequalities for special functions*, Proc. Amer. Math. Soc., **142**(9) (2014) 3059-3066, `http://arxiv.org/abs/1301.5255`.

[3] Á. Baricz, B.A. Bhayo, T.K. Pogány, *Convexity properties of generalized trigonometric and hyperbolic functions*, J. Math. Anal. Appl., **417** (2014), 244-259.

[4] M.A. Chaudhry, A. Qadir, M. Rafique, S.M. Zubair, *Extension of Euler's beta function*, J. Comput. Appl. Math., **78** (1997), 19-32.

[5] M.A. Chaudhry, A. Qadir, H.M. Srivastava, R.B. Paris, *Extended hypergeometric and confluent hypergeometric functions*, Appl. Math. Comput., **159** (2004), 589-602.

[6] A.R. Miller, *Reduction of a generalized incomplete gamma function, related Kampedé Fériet functions, and incomplete Weber integrals*, Rocky Mountain J. Math., **30** (2000), 703-714.

[7] S. R. Mondal, *Inequalities of extended beta and extended hypergeometric functions*, J. Inequal. Appl., **2017** (2017)(1):10. `http://arxiv.org/abs/1611.08423`

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