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ON THE SRIVASTAVA-SINGHAL POLYNOMIALS

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Abstract. In this study, we give new properties of the Srivastava-Singhal polynomials $G_n^{(\alpha)}(x, r, \beta, k)$. Various families of bilinear and bilateral generating functions, some special cases and several recurrence relations for these polynomials are obtained.

1. Introduction

The Srivastava-Singhal polynomials $G_n^{(\alpha)}(x, r, \beta, k)$ are defined by the generating relation (see, [11], Eq. (3.2), p. 78])

$$\sum_{n=0}^{\infty} G_n^{(\alpha)}(x, r, \beta, k)\, t^n = (1-kt)^{-\frac{\alpha}{k}} \exp\left\{\beta x^r \left[1-(1-kt)^{\frac{-r}{k}}\right]\right\} \tag{1.1}$$

where, $\alpha > -1$, k is a positive integer.

It is from (1.1) that [11],

$$\begin{aligned} G_n^{(\alpha)}(x, r, \beta, k) &= \frac{k^n}{n!} \sum_{m=0}^{\infty} \frac{(-\beta x^r)^m}{m!} \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} \left(\frac{\alpha + rj}{k}\right)_n \\ &= \frac{k^n}{n!} \sum_{j=0}^{\infty} \frac{e^{\beta x^r} (-\beta)^j}{j!} \left(\frac{\alpha + rj}{k}\right)_n x^{rj} \end{aligned} \tag{1.2}$$

where $(\lambda)_v$ denotes the Pochhammer symbol defined by

$$(\alpha)_n = \begin{cases} \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)} = \alpha(\alpha+1)...(\alpha+n-1), & n = 1, 2, 3, ... \\ (\alpha)_0 = 1. \end{cases}$$

These polynomials have the following generating relation (see, [1], pp.431):

$$\begin{aligned} &\sum_{n=0}^{\infty} \binom{n+m}{n} G_{m+n}^{(\alpha)}(x, r, \beta, k)\, t^n \\ &= (1-kt)^{\frac{-(\alpha+mk)}{k}} \exp\left\{\beta x^r \left[1-(1-kt)^{\frac{-r}{k}}\right]\right\} G_m^{(\alpha)}\left[x(1-kt)^{\frac{-1}{k}}, r, \beta, k\right] \end{aligned} \tag{1.3}$$

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In addition, we give a theorem about the addition formula for Srivastava-Singhal polynomials $G_n^{(\alpha)}(x,r,\beta,k)$:

Theorem 1.1. *We have*

$$\begin{aligned} & G_n^{(\alpha_1+\alpha_2)}(x,r,\beta_1+\beta_2,k) \\ = & \sum_{m=0}^{n} G_{n-m}^{(\alpha_1)}(x,r,\beta_1,k)\, G_m^{(\alpha_2)}(x,r,\beta_1,k)\,. \end{aligned} \tag{1.4}$$

Proof. Replacing α by $\alpha_1+\alpha_2$ and β by $\beta_1+\beta_2$ in (1.1), we obtain

$$\begin{aligned} & \sum_{n=0}^{\infty} G_n^{(\alpha_1+\alpha_2)}(x,r,\beta_1+\beta_2,k)\, t^n \\ = & (1-kt)^{-\frac{(\alpha_1+\alpha_2)}{k}} \exp\left\{(\beta_1+\beta_2)\, x^r \left[\left(1-(1-kt)^{-\left(\frac{r}{k}\right)}\right)\right]\right\} \\ = & (1-kt)^{-\frac{\alpha_1}{k}} \exp\left(\beta_1 x^r \left[1-(1-kt)^{\frac{-r}{k}}\right]\right) \\ & \times (1-kt)^{-\frac{\alpha_2}{k}} \exp\left(\beta_2 x^r \left[1-(1-kt)^{\frac{-r}{k}}\right]\right) \\ = & \sum_{n=0}^{\infty} G_n^{(\alpha_1)}(x,r,\beta_1,k)\, t^n \sum_{m=0}^{\infty} G_m^{(\alpha_2)}(x,r,\beta_2,k)\, t^m \\ = & \sum_{n=0}^{\infty}\sum_{m=0}^{n} G_{n-m}^{(\alpha_1)}(x,r,\beta_1,k)\, G_m^{(\alpha_2)}(x,r,\beta_2,k)\, t^n. \end{aligned}$$

From the coefficients of t^n on the both sides of the last equality, one can get the desired result. □

Now we recall the relationship [[12], p. 315, Eq. (83)]:

$$\begin{aligned} Y_n^{\alpha}(x;k) &= k^{-n} G_n^{(\alpha+1)}(x,1,1,k) \\ & \quad (\alpha > -1; \quad k = 1,2,...) \end{aligned} \tag{1.5}$$

where $Y_n^{\alpha}(x;k)$ denotes the Konhauser biorthogonal polynomials (cf. [13]-[17]). In particular,

$$\begin{aligned} Y_n^{\alpha}(x;1) &= L_n^{(\alpha)}(x) = G_n^{(\alpha+1)}(x,1,1,1) \\ & \quad (\alpha > -1; \quad n = 0,1,2,...) \end{aligned} \tag{1.6}$$

and the polynomials $Y_n^{\alpha}(x;2)$ were encountered earlier by Spencer and Fano [18] in certain analytical calculations involving the penetration of gamma rays through matter.

In Section 2, we prove several theorems involving various families of generating functions for the polynomials $G_n^{(\alpha)}(x,r,\beta,k)$ by applying the methot which was studied by Chen and Srivastava [10]. Further, in Section 3, as an application of these teorems, we present some generating relations for the Srivastava-Singhal polynomials $G_n^{(\alpha)}(x,r,\beta,k)$ which are given by (1.2). In the last section, we give some miscellaneous recurrence relations of the Srivastava-Singhal polynomials obtained by (1.1).

2. Bilinear and Bilateral Generating Functions

In this section, we derive several families of bilinear and bilateral generating functions for the Srivastava-Singhal polynomials $G_n^{(\alpha)}(x, r, \beta, k)$ which are generated by (1.1) and given explicitly by (1.2) using the similar method considered in (see, [2]-[9]).

We begin by stating the following theorem.

Theorem 2.1. *Corresponding to an identically non-vanishing function $\Omega_\mu(y_1, ..., y_r)$ of r complex variables $y_1, ..., y_r$ $(r \in \mathbb{N})$ and of complex order μ, ψ, let*

$$\Lambda_{\mu,\psi}(y_1, ..., y_r; \zeta) := \sum_{k=0}^{\infty} a_k \Omega_{\mu+\psi k}(y_1, ..., y_r)\zeta^k \qquad (a_k \neq 0).$$

Suppose also that

$$\begin{aligned} &\Theta_{n,p}^{\mu,\psi}(x, c, q, u; y_1, ..., y_r; \xi) \\ : &= \sum_{k=0}^{[n/p]} a_k G_{n-pk}^{(\alpha)}(x, c, q, u)\, \Omega_{\mu+\psi k}(y_1, ..., y_r)\xi^k. \end{aligned}$$

Then the bilateral generating function for $G_n^{(\alpha)}(x, c, q, u)$ is defined by

$$\begin{aligned} &\sum_{n=0}^{\infty} \Theta_{n,p}^{\mu,\psi}\left(x, c, q, u; y_1, ..., y_r; \frac{\eta}{t^p}\right) t^n \\ = \quad &(1-ut)^{\frac{-\alpha}{u}} \exp\left\{qx^c\left[1-(1-ut)^{-\frac{c}{u}}\right]\right\} \Lambda_{\mu,\psi}(y_1, ..., y_r; \eta). \end{aligned} \tag{2.1}$$

Provided that each member of (2.1) exists.

Proof. Let S denotes the left hand side of (2.1). Then we find

$$S = \sum_{n=0}^{\infty} \sum_{k=0}^{[n/p]} a_k G_{n-pk}^{(\alpha)}(x, c, q, u)\, \Omega_{\mu+\psi k}(y_1, ..., y_r)\eta^k t^{n-pk}. \tag{2.2}$$

Now, setting $n \to n + pk$ in (2.2), we obtain

$$S = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} a_k\, G_n^{(\alpha)}(x, c, q, u)\, \Omega_{\mu+\psi k}(y_1, ..., y_r)\eta^k t^n.$$

Then by the generating relation (1.1), we find

$$S = (1-ut)^{\frac{-\alpha}{u}} \exp\left\{qx^c\left[1-(1-ut)^{-\frac{c}{u}}\right]\right\} \Lambda_{\mu,\psi}(y_1, ..., y_r; \eta).$$

□

In a smilar way, we can state the following theorem.

Theorem 2.2. *Corresponding to an identically non-vanishing function $\Omega_\mu(y_1, ..., y_r)$ of r (real or complex) variables $y_1, ..., y_r$ $(r \in \mathbb{N})$ and of complex order μ, ψ, α, β let*

$$\begin{aligned} &\Lambda_{n,p,\mu,\psi}^{\alpha_1,\alpha_2}(x, c, q_1+q_2, u; y_1, ..., y_r; t) \\ : &= \sum_{k=0}^{[n/p]} a_k G_{n-pk}^{(\alpha_1+\alpha_2)}(x, c, q_1+q_2, u)\, \Omega_{\mu+\psi l}(y_1, ..., y_r)t^k, \end{aligned}$$

where $a_k \neq 0$, $n, p \in \mathbb{N}$ *and the notation* $[n/p]$ *means the greatest integer less than or equal* n/p.

Then, for $p \in \mathbb{N}$, *we have*

$$\begin{aligned} &\sum_{k=0}^{n}\sum_{l=0}^{[k/p]} a_l G_{n-k}^{(\alpha_1)}(x,c,q_1,u)\, G_{k-pl}^{(\alpha_2)}(x,c,q_2,u)\, \Omega_{\mu+\psi l}(y_1,...,y_r) z^l \\ =\; & \Lambda_{n,p,\mu,\psi}^{\alpha_1,\alpha_2}(x,c,q_1+q_2,u;y_1,...,y_r;z) \end{aligned} \tag{2.3}$$

provided that each member of (2.3) *exists.*

Proof. For convenince, let T denote the first member of the assertion (2.3) of Theorem 2.2. Then, upon subsituting for the polynomials $G_n^{(\alpha_1+\alpha_2)}(x,c,q_1+q_2,u)$ from the (1.4) into the left-hand side of (2.3), we obtain

$$\begin{aligned} T &= \sum_{k=0}^{n}\sum_{l=0}^{[k/p]} a_l G_{n-k}^{(\alpha_1)}(x,c,q_1,u)\, G_{n-pl}^{(\alpha_2)}(x,c,q_2,u)\, \Omega_{\mu+\psi l}(y_1,...,y_r) z^l \\ &= \sum_{l=0}^{[n/p]}\sum_{k=0}^{n-pl} a_l G_{n-k-pl}^{(\alpha_1)}(x,c,q_1,u)\, G_{k}^{(\alpha_2)}(x,c,q_2,u)\, \Omega_{\mu+\psi l}(y_1,...,y_r) z^l \\ &= \sum_{l=0}^{[n/p]} a_l \left(\sum_{k=0}^{n-pl} G_{n-k-pl}^{(\alpha_1)}(x,c,q_1,u)\, G_{k}^{(\alpha_2)}(x,c,q_2,u)\right) \Omega_{\mu+\psi l}(y_1,...,y_r) z^l \\ &= \sum_{l=0}^{[n/p]} a_l G_{n-pl}^{(\alpha_1+\alpha_2)}(x,c,q_1+q_2,u)\, \Omega_{\mu+\psi l}(y_1,...,y_r) z^l \\ &= \Lambda_{n,p,\mu,\psi}^{\alpha_1,\alpha_2}(x,c,q_1+q_2,u;y_1,...,y_r;z). \end{aligned}$$

□

Theorem 2.3. *Corresponding to an identically non-vanishing function* $\Omega_\mu(y_1,...,y_r)$ *of* r *complex variables* $y_1,...,y_r$ $(r \in \mathbb{N})$ *and of complex order* μ, *suppose that*

$$\Lambda_{m,p,q}[x,c,h,u;y_1,...,y_r;z] := \sum_{i=0}^{\infty} a_i G_{m+qi}^{(\alpha)}(x,c,h,u)\, \Omega_{\mu+pi}(y_1,...,y_r) z^i$$

where $a_i \neq 0$ *and*

$$\theta_{m,p,q}(y_1,...,y_r;z) := \sum_{j=0}^{[i/q]} \binom{m+i}{i-qj} a_j \Omega_{\mu+pj}(y_1,...,y_r) z^j .$$

Then, for $p, q \in \mathbb{N}$; *we have*

$$\begin{aligned} &\sum_{i=0}^{\infty} G_{i+m}^{(\alpha)}(x,c,h,u)\, \theta_{m,p,q}(y_1,...,y_r;z) t^i \\ =\; & (1-ut)^{-\left(\frac{\alpha+mu}{u}\right)} \exp\left\{hx^c\left[1-(1-ut)^{-\frac{c}{u}}\right]\right\} \\ & \times \Lambda_{m,p,q}\left(x(1-ut)^{-\frac{1}{u}}, c,h,u;y_1,...,y_r;z\left(\frac{t}{1-ut}\right)^q\right) \end{aligned} \tag{2.4}$$

provided that each member of (2.4) *exists.*

Proof. For convenience, let H denote the first member of the assertion (2.4) of Theorem 2.3. Then,

$$H=\sum_{i=0}^{\infty}G_{i+m}^{(\alpha)}\left(x,c,h,u\right)\sum_{j=0}^{[i/q]}\binom{m+i}{i-qj}a_j\Omega_{\mu+pj}(y_1,...,y_r)z^jt^i.$$

Replacing i by $i+qj$ and then using (1.3), we may write that

$$\begin{aligned}
H &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\binom{m+i+qj}{i}G_{i+m+qj}^{(\alpha)}\left(x,c,h,u\right)a_j\,\Omega_{\mu+pj}(y_1,...,y_r)z^jt^{i+qj}\\
&= \sum_{j=0}^{\infty}\left(\sum_{i=0}^{\infty}\binom{m+i+qj}{i}G_{i+m+qj}^{(\alpha)}\left(x,c,h,u\right)t^i\right)a_j\Omega_{\mu+pj}(y_1,...,y_r)(zt^q)^j\\
&= \sum_{j=0}^{\infty}a_j(1-ut)^{\frac{-(\alpha+qju+mu)}{u}}\exp\left\{hx^c\left[1-(1-ut)^{-\frac{c}{u}}\right]\right\}\\
&\quad\times G_{m+qj}^{(\alpha)}\left[x\left(1-ut\right)^{\frac{-1}{u}},c,h,u\right]\Omega_{\mu+pj}(y_1,...,y_r)(zt^q)^j\\
&= (1-ut)^{\frac{-(\alpha+mu)}{u}}\exp\left\{hx^c\left[1-(1-ut)^{\frac{-c}{u}}\right]\right\}\\
&\quad\times\sum_{j=0}^{\infty}a_jG_{m+qj}^{(\alpha)}\left[x\left(1-ut\right)^{-\frac{1}{u}}c,h,u\right]\Omega_{\mu+pj}(y_1,...,y_r)\left(\frac{zt^q}{(1-ut)^q}\right)^j\\
&= (1-ut)^{\frac{-(\alpha+mu)}{u}}\exp\left\{hx^c\left[1-(1-ut)^{-\frac{c}{u}}\right]\right\}\\
&\quad\times\Lambda_{m,p,q}\left(x\left(1-ut\right)^{-\frac{1}{u}},c,h,u;y_1,...,y_r;z\left(\frac{t}{1-ut}\right)^q\right),
\end{aligned}$$

which completes the proof. □

3. Special Cases

When the multivariable function $\Omega_{\mu+\psi k}(y_1,...,y_r)$, $k\in\mathbb{N}_0$, $r\in\mathbb{N}$, is expressed in terms of simpler functions of one and more variables, then we can give further applications of the above theorems. We first set

$$\Omega_{\mu+\psi k}(y_1,...,y_r)=h_{\mu+\psi k}^{(\alpha_1,\alpha_2,...,\alpha_r)}(y_1,...,y_r)$$

in Theorem 2.1 , where the multivariable extension of the Lagrange-Hermite polynomials $h_{\mu+\psi k}^{(\alpha_1,\alpha_2,...,\alpha_r)}(x_1,...,x_r)$ [2], generated by

$$\begin{gathered}\prod_{j=1}^{r}\left\{(1-x_jt^j)^{-\alpha_j}\right\}=\sum_{n=0}^{\infty}h_n^{(\alpha_1,...,\alpha_r)}\left(x_1,...,x_r\right)t^n,\\ \left(\alpha\in\mathbb{C}\ ;\ |t|<\min\left\{|x_1|^{-1},|x_2|^{-1/2},...,|x_r|^{-1/r}\right\}\right).\end{gathered}\tag{3.1}$$

We are thus led to the following result which provides a class of bilateral generating functions for the multivariable extension of the Lagrange-Hermite polynomials $h_{\mu+\psi k}^{(\alpha_1,\alpha_2,...,\alpha_r)}(y_1,...,y_r)$ and the Srivastava-Singhal polynomials.

Corollary 3.1. *If*

$$\Lambda_{\mu,\psi}(y_1,...,y_r;\zeta) \quad : \quad = \sum_{k=0}^{\infty} a_k h_{\mu+\psi k}^{(\alpha_1,\alpha_2,...,\alpha_r)}(y_1,...,y_r)\zeta^k$$
$$(a_k \neq 0\,, \quad \mu,\psi \in \mathbb{C})\,,$$

then, we have

$$\sum_{n=0}^{\infty}\sum_{k=0}^{[n/p]} a_k G_{n-pk}^{(\alpha)}(x,c,q,u)\, h_{\mu+\psi k}^{(\alpha_1,\alpha_2,...,\alpha_r)}(y_1,...,y_r)\frac{\zeta^k}{t^{pk}}t^n$$
$$= \quad (1-ut)^{\frac{-\alpha}{u}} \exp\left\{qx^c\left[1-(1-ut)^{-\frac{c}{u}}\right]\right\}\Lambda_{\mu,\psi}(y_1,...,y_r;\zeta).$$

Remark 3.1. Using the generating relation (3.1) for the multivariable polynomials $h_{\mu+\psi k}^{(\alpha_1,\alpha_2,...,\alpha_r)}(x_1,...,x_r)$ and getting $a_k = 1$, $\mu = 0$, $\psi = 1$ in Corollary 3.1, we find that

$$\sum_{n=0}^{\infty}\sum_{k=0}^{[n/p]} G_{n-pk}^{(\alpha)}(x,c,q,u)\, h_{k}^{(\alpha_1,\alpha_2,...,\alpha_r)}(y_1,...,y_r)\zeta^k t^{n-pk}$$
$$= \quad (1-ut)^{\frac{-\alpha}{u}} \exp\left\{qx^c\left[1-(1-ut)^{-\frac{c}{u}}\right]\right\}\prod_{j=1}^{r}\left\{(1-y_j\zeta^j)^{-\alpha_j}\right\}.$$

$$\left(|\zeta| < \min\left\{|y_1|^{-1}, |y_2|^{-1/2}, ..., |y_r|^{-1/r}\right\}, \qquad |t| < 1\right)$$

If we set $r = 1$, $y_1 = y$ and

$$\Omega_{\mu+\psi k}(y) = G_{\mu+\psi k}^{(\alpha_3)}(y,c,q_3,w)$$

in Theorem 2.2, we have the following bilinear generating functions for the Srivastava-Singhal polynomials.

Corollary 3.2. *Let*

$$\Lambda_{n,p,\mu,\psi}^{\alpha_1,\alpha_2}(x,c,q_1+q_2,u;y,c,q_3,w;t)$$
$$: \quad = \sum_{k=0}^{[n/p]} a_k G_{n-pk}^{(\alpha_1+\alpha_2)}(x,c,q_1+q_2,u)\, G_{\mu+\psi k}^{(\alpha_3)}(y,c,q_3,w)\, t^k$$

where $a_k \neq 0\,, \mu,\psi \in \mathbb{C}$. *Then, we get*

$$\sum_{k=0}^{n}\sum_{l=0}^{[k/p]} a_l G_{n-k}^{(\alpha_1)}(x,c,q_1,u)\, G_{k-pl}^{(\alpha_2)}(x,c,q_2,u)\, G_{\mu+\psi l}^{(\alpha_3)}(y,c,q_3,w)\, z^l$$
$$= \quad \Lambda_{n,p,\mu,\psi}^{\alpha_1,\alpha_2}(x,c,q_1+q_2,u;y,c,q_3,w;z) \tag{3.2}$$

provided that each member of (3.2) *exists.*

Remark 3.2. If we take $a_l = 1$, $\mu = 0$, $\psi = 1$, $p = 1$, $z = 1$, $x = y$, $u = w$ and then use the relation (1.4) for Srivasatava-Singhal polynomials in Corollary 3.2, we have

$$\begin{aligned}
&\sum_{k=0}^{n}\sum_{l=0}^{k} G_{n-k}^{(\alpha_1)}(x,c,q_1,u)\, G_{k-l}^{(\alpha_2)}(x,c,q_2,u)\, G_{l}^{(\alpha_3)}(x,c,q_3,u) \\
=\;& \sum_{k=0}^{n} G_{n-k}^{(\alpha_1)}(x,c,q_1,u) \sum_{l=0}^{k} G_{k-l}^{(\alpha_2)}(x,c,q_2,u)\, G_{l}^{(\alpha_3)}(x,c,q_3,u) \\
=\;& \sum_{k=0}^{n} G_{n-k}^{(\alpha_1)}(x,c,q_1,u)\, G_{k}^{(\alpha_2+\alpha_3)}(x,c,q_2+q_3,u) \\
=\;& G_{n}^{(\alpha_1+\alpha_2+\alpha_3)}(x,c,q_1+q_2+q_3,u)\,.
\end{aligned}$$

If we set

$$r = 1\ ,\ y_1 = y \ \text{ and } \Omega_{\mu+\psi k}(y) = g_n^{(s)}(\lambda, y)$$

in Theorem 2.3, where the generalized Cesàro polynomials $g_n^{(s)}(\lambda, x)$ is generated by [6],

$$\sum_{n=0}^{\infty} g_n^{(s)}(\lambda, x) t^n = (1-t)^{-s-1}(1-xt)^{-\lambda}. \tag{3.3}$$

Thus, we get a family of the bilateral generating functions for the generalized Cesàro polynomials and the Srivastava-Singhal polynomials as follows:

Corollary 3.3. *If*

$$\Lambda_{m,p,q}[x,c,h,u;\lambda,y;z] \;:=\; \sum_{i=0}^{\infty} a_i G_{m+qi}^{(\alpha)}(x,c,h,u)\, g_{\mu+pi}^{(s)}(\lambda,y) z^i$$
$$(a_i \neq 0,\ m \in \mathbb{N}_0,\ \mu,\psi \in \mathbb{C})$$

and

$$\theta_{m,p,q}(\lambda, y; z) := \sum_{j=0}^{[i/q]} \binom{m+i}{i-qj} a_j g_j(\lambda, y) z^j$$

where $p, q \in \mathbb{N}$, *then we have*

$$\begin{aligned}
&\sum_{i=0}^{\infty} G_{i+m}^{(\alpha)}(x,c,h,u)\, \theta_{m,p,q}(\lambda,y;z) t^i \\
=\;& (1-ut)^{\frac{-(\alpha+mu)}{u}} \exp\left\{hx^c\left[1-(1-ut)^{-\frac{c}{u}}\right]\right\} \\
&\times \Lambda_{m,p,q}\left(x(1-ut)^{-\frac{1}{u}}, c,h,u;\lambda,y;z\left(\frac{t}{1-ut}\right)^q\right)
\end{aligned} \tag{3.4}$$

provided that each member of (3.4) *exists.*

Notice that, for every suitable choice of the coefficients a_k ($k \in \mathbb{N}_0$), if the multivariable functions $\Omega_{\mu+\psi k}(y_1, ..., y_r)$, $r \in \mathbb{N}$, are expressed as an appropriate product of several simpler relatively functions, the assertions of Theorem 2.1, 2.2 , 2.3 can be applied to yield many different families of multilinear and multilateral generating functions for the Srivastava-Singhal polynomials.

4. Recurrence Relations

We now discuss some miscellaneous recurrence relations of the Srivastava-Singhal polynomials $G_n^{(\alpha)}(x,r,\beta,k)$ given by (1.1). By differentiating each member of the generating function relation (1.1) with respect to x and using

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}A(k,n)=\sum_{n=0}^{\infty}\sum_{k=0}^{n}A(k,n-k),$$

we arrive at the following (differential) recurrence relation for the Srivastava-Singhal polynomials $G_n^{(\alpha)}(x,r,\beta,k)$ given explicitly by (1.1):

$$\frac{\partial}{\partial x}G_n^{(\alpha)}(x,r,\beta,k)=\beta r x^{r-1}G_n^{(\alpha)}(x,r,\beta,k)-\beta r x^{r-1}\sum_{m=0}^{n}\left(\frac{r}{k}\right)_m\frac{k^m}{m!}G_{n-m}^{(\alpha)}(x,r,\beta,k). \quad (4.1)$$

Using the relation (1.5) and getting $\alpha\to\alpha+1$, $r=\beta=k=1$ in (4.1), we have the following (differential) recurrence relation for the Laguerre polynomials:

$$\frac{d}{dx}L_n^{(\alpha)}(x)=L_n^{(\alpha)}(x)-\sum_{m=0}^{n}L_{n-m}^{(\alpha)}(x).$$

Similarly, the special case $\alpha\to\alpha+1$, $r=\beta=1$ of (4.1), we have the following (differential) recurrence relation for the biorthogonal Konhauser polynomials:

$$\frac{d}{dx}Y_n^{\alpha}(x;k)=Y_n^{\alpha}(x;k)-\sum_{m=0}^{n}\left(\frac{1}{k}\right)_m\frac{1}{m!}Y_{n-m}^{\alpha}(x;k).$$

By differentiating each member of the generating function relation (1.1) with respect to t, we have the following another recurrence relation for these polynomials:

$$(n+1)G_{n+1}^{(\alpha)}(x,r,\beta,k)=\alpha\sum_{m=0}^{n}k^mG_{n-m}^{(\alpha)}(x,r,\beta,k)-\beta r x^{r}\sum_{p=0}^{n}\left(\frac{r}{k}+1\right)_p\frac{k^p}{p!}G_{n-p}^{(\alpha)}(x,r,\beta,k). \quad (4.2)$$

Choosing $\alpha\to\alpha+1$, $r=\beta=k=1$ in (4.2), we have the following recurrence relation for the Laguerre polynomials:

$$(n+1)L_{n+1}^{(\alpha+1)}(x)=(\alpha+1)\sum_{m=0}^{n}L_{n-m}^{(\alpha+1)}(x)-x\sum_{p=0}^{n}(p+1)\,L_{n-p}^{(\alpha+1)}(x).$$

Writing p instead of m; we may write that

$$(n+1)L_{n+1}^{(\alpha+1)}(x)=\sum_{p=0}^{n}(\alpha+1-xp-x)L_{n-p}^{(\alpha+1)}(x).$$

Finally, setting $\alpha\to\alpha+1$, $r=\beta=1$ in (4.2), we can state the following recurrence relation for the biorthogonal Konhauser polynomials:

$$(n+1)Y_{n+1}^{\alpha+1}(x;k)=(\alpha+1)\sum_{m=0}^{n}k^mY_{n-m}^{\alpha+1}(x;k)-x\sum_{p=0}^{n}\left(\frac{1}{k}+1\right)_p\frac{k^p}{p!}Y_{n-p}^{\alpha+1}(x;k).$$

Writing p instead of m; we may write that

$$(n+1)Y_{n+1}^{\alpha+1}(x;k)=\sum_{p=0}^{n}k^{p}\left((\alpha+1)-\frac{x}{p!}\left(\frac{k+1}{k}\right)_{p}\right)Y_{n-m}^{\alpha+1}(x;k)\,.$$

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